APMAE3102: Applied Mathematics II
Lecture Notes

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1 Introduction

Some general remarks on PDEs:

- usually hard to solve, and few are completely understood
- non-trivial numerical solutions

PDEs involve:

- equation of partial derivatives
- unknown functions of **more than one variable** (as contrast to ODE systems which could have several functions of **single variable**).

1.1 Brief Review of ODE

A typical **initial value problem in ODE** looks like:

$$\begin{cases} \frac{d}{dt}\vec{x}' = \vec{x}' = f(x,t) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

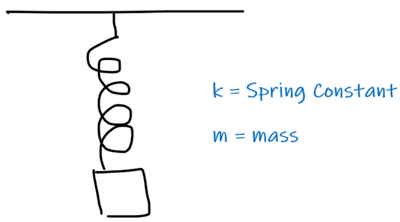
where:

- this would be a **first order ODE** because the highest order of derivative is 1
- $\vec{x} = \begin{bmatrix} x_1(t) \\ ... \\ x_n(t) \end{bmatrix}$, all of which are **single variable** functions
- the initial condition $\vec{x}_0$ (an n-dimensional vector) uniquely determines the solution for $t > 0$.

Note:

> All of which will be different with an PDE, which you will see later.

Consider the example of a spring with mass:



***Example*: A simple ODE**

In the above example, we can simply write down the ODE (ignoring the damping):

$$my'' = -ky \tag{1}$$

with the initial condition:

$$\begin{cases} y(0) = y_0 \\ y'(0) = v_0 \end{cases}$$

to solve it using ODE approaches, we can do:

Let $\vec{x} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}$, then we have:

$$\begin{aligned} \frac{d}{dt}\vec{x}(t) &= \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} \\ &= \begin{bmatrix} y' \\ -\frac{k}{m}y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \vec{x} \end{aligned}$$

Now we have:

$$\frac{d}{dt}\vec{x} = A\vec{x} \tag{2}$$

where:

- A is the constant matrix
- we have successfully converted the equation back to "single variable" of only $\vec{x}$, which becomes solvable

With the initial condition of $\vec{x}_0$, and that A is constant, we know the **unique solution** is:

$$\vec{x} = e^{At}\vec{x}_0 \tag{3}$$

Reminder:

> - *the exponential of a matrix can be basically understood as:*
>
> $$e^{At} = \lim_{n\to\infty} \frac{t^n}{n!} A^n$$
>
> - *the proof of the unique solution of the above ODE is, in short:*
> *Consider the quantity* $\frac{d}{dt}\left(e^{-At}\vec{x}(t)\right)$
>
> $$\frac{d}{dt}\left(e^{-At}\vec{x}(t)\right) = e^{-At}\vec{x}(t)' - e^{At}A\vec{x} = 0$$
>
> *since* $\vec{x}(t)' = A\vec{x}$
> *This means that the function:*
>
> $$t \to e^{-At}\vec{x}(t)$$
>
> *is constant in time. This yields:*
>
> $$\vec{x}(t) = e^{At}\vec{x}_0$$

1.2 Brief Review of Linear Algebra

Consider the same ODE system:

$$\vec{x}(t)' = A\vec{x}(t)$$

and consider the equation:

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

where:

- $\vec{v}$ is just a constant vector

Substituting the equation into the ODE, we get:

$$A\vec{v} = \lambda\vec{v}$$

where:

- this means that for the system $\vec{x} = e^{\lambda t}\vec{v}$, any eigenvalue/eigenvector pair for $\lambda, \vec{v}$ will solve the ODE given

Additionally, suppose the matrix A is symmetric.

Reminder:

> *For a symmetric matrix $A = A^T$, it has the following properties:*
>
> - *A has exactly n eigenvalues (not necessarily distinct)*
> - *A has a complete set (n) of orthogonal eigenvectors*

Therefore, the **complete** solution of the above becomes:

$$\vec{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \vec{v}_i \tag{4}$$

where:

- $\lambda_i, \vec{v}_i$ are eigenvalue, eigenvector pairs
- c_i is solved by the initial condition $\vec{x}_0 = \sum_{i=1}^{n} c_i \vec{v}_i$

Note:

> In general, it is hard to solve the equation $\vec{x}_0 = \sum_{i=1}^{n} c_i \vec{v}_i$, but since we have a symmetric matrix A, we know that we have orthogonal eigenvector v_i. As a result:
>
> $$\begin{cases} \vec{x}_0 \cdot \vec{v}_1 = c_1 \vec{v}_1 \cdot \vec{v}_1, \quad c_1 = \frac{\vec{x}_0 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \\ \dots \end{cases}$$

1.3 Introduction to PDE

First of all, some **notations**:

1. PDE involves functions of more than one variable, and a typical symbol for that function is, for example:

$$u(x, t)$$

2. the below notations are all equivalent:

$$\frac{\partial u}{\partial t} \equiv u_t$$
$$\frac{\partial^2 u}{\partial t^2} \equiv u_{tt}.$$

An example of PDE would be perhaps better.

***Example*: Advection Equation**

> Advection describes the transport of something via fluid flow. The equation is:
>
> $$\rho_t + u_0 \rho_x = 0 \tag{5}$$
>
> or equivalently:
>
> $$\frac{\partial \rho}{\partial t} + u_0 \frac{\partial \rho}{\partial x} = 0$$
>
> where:
>
> - $\rho(x, t)$ describes some density of particle as a function of position and time, u_0 will be a constant (unit of velocity)

In the above equation 5, the solution is in any differentiable function f, such that

$$\rho(x,t) = f(x - u_0 t)$$

where:

- this type of solution is also called a **classical solution**

Proof. Simply substituting the function in, we get:

$$\begin{aligned}\frac{\partial}{\partial t} f(x-u_0t) + u_0 \frac{\partial}{\partial x} f(x-u_0t) &= -u_0 f(x-u_0t)' + u_0 f(x-u_0t)' \\ &= 0.\end{aligned}$$

□

Reminder:

The chain rule of partial derivatives goes like:

- *for a function $z = f(x_{(s,t)}, y_{(s,t)})$, we have:*

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

  where:

  - *if we have $x = x(s), y = y(s)$ instead, then it would be a total derivative, i.e. $\frac{dz}{ds}$ on LHS.*

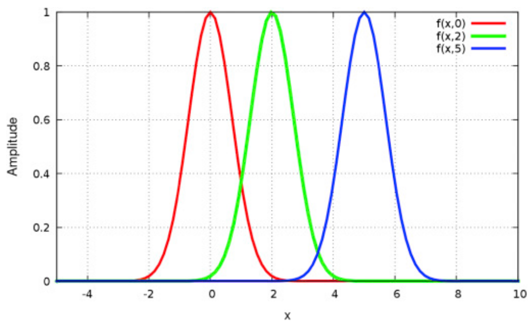
- *for the above example, we have:*

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial t} = \frac{df}{dt} = -u_0 \frac{df}{d(x-u_0t)}$$

Visualizing the addiction equation, suppose:

$$\rho(x,t) = e^{-(x-u_0t)^2}$$

taking $u_0 = +1$, we see:



where:

- the density ρ basically moves to the right at the rate of 1

Note:

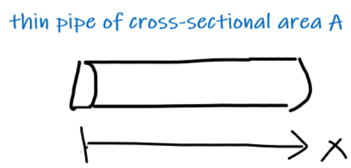
The take away message here is that:

- the initial condition here for a PDE is a **function**, i.e. $\rho(x,0) = \rho_0(x)$. (Think of this like a snapshot of the configuration of $\rho(x,t)$ at $t=0$, which has two variables).
- the solution depends continuously on the initial "function"

2 Heat Equation

2.1 Modelling

Again, consider the density in 1-D: $\rho = \rho(x, t)$:



and that the particles travel in the pipe with speed $u(x, t)$.

Considering a section of the pipe, $a < x < b$, we can calculate the number of particles inside that volume section:

$$N(a < x < b, t) = \int_a^b \rho(x, t) A \, dx \tag{6}$$

Suppose particles in that pipe is created (e.g. via chemical reactions) at a rate of $Q(x, t)$ (could be in unit of density per unit volume).

Then, to describe what is happening in that section of pipe, we can use the conservation of particles:

$$\frac{d}{dt} N(a < x < b, t) = \int_a^b Q(x, t) A \, dx + A\rho(a, t)u(a, t) - A\rho(b, t)u(b, t)$$
$$\int_a^b \frac{\partial \rho}{\partial t} A \, dx = \int_a^b QA \, dx - \int_a^b \frac{\partial}{\partial x}(\rho u) A \, dx.$$

Therefore, combining the RHS and doing the cancellations:

$$\frac{\partial \rho}{\partial t} = Q - \frac{\partial}{\partial x}(\rho u)$$

or equivalently:

$$\rho_t + (\rho u)_x = Q$$

Note:

Here, in 1-D, we used the result of the fundemental theorem of calculus, such that:

$$\int_a^b \frac{\partial}{\partial x} f(x, t) \, dx = f(b, t) - f(a, t)$$

For a higher dimension, the **gradient theorem** might come in handy:

$$\int_C \nabla f \cdot d\vec{r} = f(r_{(b)}) - f(r_{(a)}) \tag{7}$$

Hence, from the above derivation, we have:

Theorem 2.1: Modelling in 1-D

For $\rho(x,t)$ being the density of particles in a thin pipe moving with velocity $u = u(x,t)$, and $Q(x,t)$ being the rate of creation of particles, we have the **modelling equation**:

$$\rho_t + (\rho u)_x = Q \tag{8}$$

where:

- advection equation just has $Q = 0$, $u = u_0$.

Note:

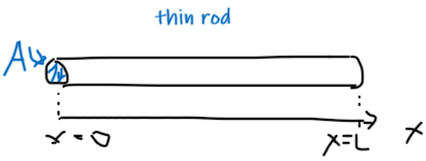
At this point, we haven't talked about (later in the course):

- how to solve such an equation
- what initial conditions are needed
- existence of solutions

2.2 Heat/Diffusion Equation

Another type of equation frequently used is the heat equation.

For 1-D, the setup is simple. Consider **heat flow through the following material**:



Let $e(x,t)$ denote the thermal energy density, and $\varphi(x,t)$ denot the heat flux form left to right. Then, we have:

$$\int_a^b e(x,t) A\, dx = \text{Thermal Energy between } a < x < b$$

Now, consider a source that generates/loses thermal energy with $Q(x,t)$ (heat energy density per unit time). Applying the **conservation of energy** (again, conservation law):

$$\begin{aligned}\int_a^b \frac{\partial}{\partial t} e(x,t) A\, dx &= \int_a^b Q(x,t) A\, dx + A\varphi(a,t) - A\varphi(b,t) \\ &= \int_a^b QA\, dx - \int_a^b \frac{\partial}{\partial x}\varphi(x,t) A\, dx.\end{aligned}$$

Hence, we get the same thing as the modelling equation:

$$\frac{\partial}{\partial t}e(x,t) = Q - \frac{\partial}{\partial x}\varphi(x,t)$$

However, this does not suffice for our puporse, because:

- for a typical application, **only Q will be known**

Therefore, we need to consider the relationship between $e(x,t)$ and $\varphi(x,t)$.

Reminder:

> *In thermodynaimcs, we know the following:*
>
> 1. *the thermal energy density $e(x,t)$:*
>
> $$e(x,t) = c(x)\rho(x)u(x,t) \tag{9}$$
>
> *where:*
>
> - *$c(x)$ denotes specific heat, $\rho(x)$ denotes density, and $u(x,t)$ denotes temperature distribution*
> - *they are a function of x because the material might not be uniform*
>
> 2. *the heat flux was:*
>
> $$\varphi(x,t) = -K\vec{\nabla}u(x,t) = -K\frac{\partial u}{\partial x} \tag{10}$$
>
> *where:*
>
> - *K represents heat conductivity (larger K means greater conductivity)*
> - *$K = K(x)$ if the material is in inhomogeneous*
> - *$u(x,t)$ represents the temperature distribution*

Now, substituting equation (9) and (10), we have:

$$c(x)\rho(x)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right) = Q$$

and if the material is homogeneous, and that $Q(x,t) = 0$, we have the easier heat equation:

$$\begin{aligned} c\rho\frac{\partial u}{\partial t} - K_0\frac{\partial^2 u}{\partial x^2} &= Q \\ u_t - \frac{K_0}{c\rho}u_{xx} &= 0 \\ u_t &= \mathrm{K}u_{xx}. \end{aligned}$$

where:

- $\mathrm{K} \equiv \frac{K_0}{c\rho}$

Hence, we have the heat/diffusion equation:

Theorem 2.2: Heat/Diffusion Equation

For general material:

$$c(x)\rho(x)\frac{\partial}{\partial t}u(x,t) = Q(x,t) + \frac{\partial}{\partial x}\left(K(x)\frac{\partial}{\partial x}u(x,t)\right) \tag{11}$$

where:

- $c(x)$ denotes specific heat, $\rho(x)$ denotes density, $Q(x,t)$ denotes heat density generated/loss by a source, and $u(x,t)$ denotes temperature distribution

For a **homogenous matieral with no** Q, we have:

$$u_t = \mathrm{K}u_{xx} \tag{12}$$

where:

- $\mathrm{K} = \frac{K_0}{c\rho}$ being a constant

Reminder:

In thermodynamics course, we also derived the diffusion equation by considering a small sphere and using Gauss' Theorem.

First, consider a flow of Γ *for a sphere* S*. Using the conservation theorem:*

$$\oint_S \vec{\Gamma}\cdot d\vec{A} = -\frac{d}{dt}\int n dV$$

then applying the divergence theorem:

$$\vec{\nabla}\cdot\vec{\Gamma} = -\frac{dn}{dt}$$

now, plugging in the thermodynamics quantity of flux $\vec{\Gamma} = -D\vec{\nabla}n$*:*

$$\vec{\nabla}\cdot\left(-D\vec{\nabla}n\right) = -\frac{dn}{dt}$$

for a constant D*, we have the simple case of diffusion/heat equation as well:*

$$D\vec{\nabla}^2 n = \frac{dn}{dt} \tag{13}$$

Now, for a 1-D system, we have:

$$u_t = \mathrm{K}u_{xx}$$

but what initial condtions/boundary conditions do we need to solve it?

Corollary 2.1

To uniquely solve the above equation (equation (12)), we need:

- one initial condition (describing t, due to **first order time derivative**)
  - for example $u(x, 0) = u_0(x)$
- two boundary conditinos (describing x, due to the **second order partial space derivative**)
  - see the next section

2.2.1 Boundary Conditions of Heat Equation

From ODE, we know there are a couple different forms of boundary conditions:

1. Dirichlet Boundary Condition
   - for example:
   $$u(0, t) = u_B(t)$$
   where B stands for a "bath" (i.e. a large reseirvoir whose own temperature does not change on contact)
2. Neumann's boundary condition:
   - for example:
   $$-K_0 \left. \frac{du}{dx} \right|_{(0,t)} = \varphi(0, t) = \varphi_0(t)$$
   so that if $\varphi = 0$, it means it is a good insulator
3. Newtonian cooling condition:
   - for example:
   $$-K_0 \left. \frac{\partial u}{\partial x} \right|_{L,t} = -H \left[u(L, t) - u_B(t) \right]$$
   where:
     - $H > 0$ is the heat transfer coefficient
     - the above would represent the case where some reseirvoir (e.g. cool air) carrying energy from the object (e.g. a hot rod) away
     - $H = 0$ would again signify a perfect insulator

2.2.2 Steady State Solutions

These classes of solutions consider the case where the material has reached a steady state, so that the time derivative would be 0.

In general, the steady state solution of:

$$0 = \mathrm{K}u_{xx}$$
$$u(x,t) = c_1 x + c_2.$$

takes **different forms depending on the initial/boundary conditions**.

***Example*: Dirichlet Steady State**

Using Dirichlet's boundary condition, we have:

$$\begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases}$$

Substituting into the solution of $u(x,t)$:

$$\begin{cases} u(0) = c_2 = T_1 \\ u(L) = c_1 L + T_1 = T_2 \end{cases}$$

Hence, we obtain the solution:

$$u(x,t) = \frac{T_2 - T_1}{L} x + T_1$$

for $0 \le x \le L$, and it means:

- **a material reaching steady state with Dirichlet's Boundary Condition will have a constant temperature gradient distribution** (which makes sense)

***Example*: Neumann Steady State**

This is slightly more complicated. Consider Neumann's boundary condition:

$$\begin{cases} u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{cases}$$

which indicates an insulator at both ends already

substituting into the $u(x,t)$ equation, we only get:

$$u_x = c_1 = 0$$

Hence we only have, **at steady state** :

$$u(x,t) = c_2$$

and apparently something is missing.

Suppose we are **not at a steady state yet** :

$$\begin{aligned} u_t &= K u_{xx} \\ \int_0^L \frac{\partial}{\partial t} u(x,t)\,dx &= \int_0^L K u_{xx}\,dx \\ \frac{d}{dt}\int_0^L u(x,t)\,dx &= K\left(u_x(L,t) - u_x(0,t)\right). \end{aligned}$$

Therefore, substituting the Neumann's boundary condition, we know:

$$\frac{d}{dt}\int_0^L u(x,t)\,dx = 0$$

hence:

- $\int_0^L u(x,t)\,dx$ is constant **for all time (including the time before a steady state)**

This means that:

$$\int_0^L u(x,t)\,dx = \int_0^L u(x,0)\,dx = c_2 L$$

Hence, since the above solves c_2, we conclude with:

$$u(x,t) = \frac{1}{L}\int_0^L u(x,0)\,dx$$

notice that $u(x,0)$ is the **initial condition**, and it means that:

- for an completely insulated material, the steady state will be **temperature/thermal energy being uniformly distributed, with the total energy being the same as the initial total energy**

Note:

> Different from the Dirichlet's boundary condition, solving for the steady state required the additional initial condition here.
>
> This is because, in fact, the form of the steady state solution **itself already is part of the Neumann's boundary condition, imposing $u_x(0,t) = u_x(L,t)$ since it is linear**. As a result, we needed one more condition to solve for the steady state.

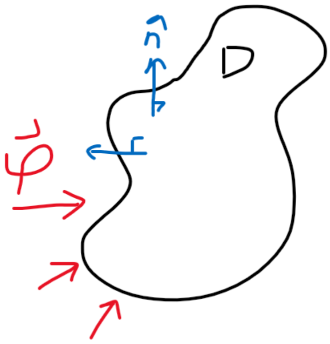
2.2.3 3D Heat Equation

To convert the heat equation into 3-D, we basically consider $u(x, y, z, t)$, and then consider

Heat flux in 3-D:

$$\vec{\varphi} = -\mathrm{K}\vec{\nabla}u$$

and then consider a 3-D volume:



Hence we have:

$$\begin{aligned}\frac{\partial}{\partial t}\iiint_D u dV &= \oiint -\vec{\varphi}\cdot d\vec{S} \\ \iiint \frac{\partial u}{\partial t} dV &= \iiint \vec{\nabla}\cdot\left(K\vec{\nabla}u\right) dV \\ u_t &= K_0\nabla^2 u.\end{aligned}$$

where:

- it assumes a system of no source $Q = 0$, and a uniform material $K = K_0$

Theorem 2.3: 3D Heat Equation

For a uniform material and no source, the heat equation in 3D is:

$$u_t = K_0\nabla^2 u \tag{14}$$

where:

- ∇^2 is called the Laplacian

2.3 Linearity and Homogeneity of Heat Equation

First, we want to define what it means to be linear and homogeneous.

Definition 2.1: Operator Linearity

An operator $\mathcal{L}$ is **linear** if it satisfies:

$$\mathcal{L}[c_1 u_1 + c_2 u_2] = c_1 \mathcal{L} u_1 + c_2 \mathcal{L} u_2 \tag{15}$$

where:

- u_1, u_2 are functions, c_1, c_2 are arbitrary constants

Some examples of unknown functions include:

Linear Algebra (A being the linear operator):

$$A(c_1 \vec{x} + c_2.\vec{y}) = c_1 A\vec{x} + c_2 A\vec{y}$$

more importantly, derivatives as well:

$$\mathcal{L}[y(t)] = y'' + p(t)y' + q(t)y$$

Corollary 2.2: Linearity of Heat Equation

For the following heat equation:

$$u_t = ku_{xx} + Q$$

we can write it as a linear operator:

$$(u_t - ku_{xx}) = Q$$

$$\mathcal{L}[u] = u_t - ku_{xx} = Q \tag{16}$$

where:

- $\mathcal{L}$ is linear because:

$$\begin{aligned}\mathcal{L}[u+v] &= (u+v)_t - k(u+v)_{xx} \\ &= u_t - ku_{xx} + v_t - kv_{xx} \\ &= \mathcal{L}[u] + \mathcal{L}[v].\end{aligned}$$

and additionally:

Corollary 2.3: Linearity of Boundar Conditions

The boundary conditions of the **Heat equation (in 1-D)** were:

1. Dirichlet: $u(0,t) = T_1(t)$

2. Neumann: $u_x(0,t) = \varphi(t)$
3. Newtonian: $-K_0 u_x(0,t) = -H[u(0,t) - u_B(t)]$

and **all of them can be written as**:

$$\beta_1 u(0,t) + \beta_2 u_x(0,t) = g(t)$$

Hence, the boundary condition can also be a **linear operator:**

$$\mathcal{B}[u] = \beta_1 u(0,t) + \beta_2 u_x\,(0,t) = g(t) \tag{17}$$

Corollary 2.4: Heat Equation using Linear Operators

From the previous two corollaries, we can re-write the heat equation as:

$$\begin{cases} \mathcal{L}[u] = Q \\ \mathcal{B}[u] = g(t) \end{cases} \tag{18}$$

where:

- $\mathcal{B}[u]$ is assumed to be evaluated at $x = 0, L$
- the initial conditions are not shown here

Now, we go to homogeneity:

Definition 2.2: Homogeneity of Heat Equation

The corollary 2.3 is also homogeneous if:

$$\begin{cases} \mathcal{L}[u] = 0 \\ \mathcal{B}[u] = 0 \end{cases} \tag{19}$$

so that $u = 0$ is **always** a trivial solution to the system

2.3.1 Superposition Principle

Theorem 2.4: Superposition of Heat Equation

The **two solution of the *same heat equation/system*** can be superposed if:

$$\mathcal{L}[u_1] = \mathcal{L}[u_2] = 0$$

so that:

$$u = c_1 u_1 + c_2 u_2$$

is also a solution.

Note:

> Since they are solving the same system, they solve/have the same boundary condition as well.

2.4 Separable Heat Equation

Theorem 2.5: Separable Heat Equation

The **solution** of heat equation is separable if **it is linear and homogeneous**, such that:

$$\begin{cases} \text{PDE becomes:} & u_t = ku_{xx} \\ \text{B.C. becomes:} & u(0,t) = u(L,t) = 0 \\ \text{I.C. is still:} & u(x,0) = f(x) \end{cases}$$

then the solution of the heat equation can be **written in the form:**

$$u(x,t) = \psi(x)G(t) \tag{20}$$

and all superposition of solutions will satisfy both the PDE and the boundary conditions

Derivation: Solving Separable Heat Equation

Again, suppose with an Ansatz that $u = \psi(x)G(t)$. We have:

$$\begin{aligned} u_t &= ku_{xx} \\ \psi(x)G'(t) &= k\psi''(x)G(t) \\ \frac{1}{k}\frac{G'(t)}{G(t)} &= \frac{\psi''(x)}{\psi(x)}. \end{aligned}$$

hence we get:

$$\frac{1}{k}\frac{G'(t)}{G(t)} = \frac{\psi''(x)}{\psi(x)} = -\lambda$$

so we get two **ODE**:

$$\begin{cases} G'(t) = -k\lambda G(t), & G(t) = ce^{-k\lambda t} \\ \psi''(x) = -\lambda\psi(x), & \psi(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x \end{cases}$$

Note:

> The above gives a **negative sign for lambda** (assuming positive lambda), because we know that $u \to 0$ as $t \to \infty$, hence the equation for G would make sense to $G \to 0$.
>
> Also, the above solutions **assumes** that $\lambda > 0$.
>
> 1. if $\lambda = 0$, we would have $\psi = c_1 x + c_2$. Then, solving with the boundary condition, we get $\psi = 0$, which becomes a trivial solution
>
> 2. if $\lambda < 0$, then we would have $G \to \infty$ as time increases. Hence, it forces $\psi = 0$ or $G = 0$, which again becomes a trivial solution for u.

Last but not least, we solve the value λ **using the boundary conditions:**

$$\begin{cases} \psi(0) = c_2 = 0 \\ \psi(L) = c_1 \sin \sqrt{\lambda} L = 0 \end{cases}$$

Therefore, we get:

$$\sin \sqrt{\lambda} L = 0$$

$$\lambda = \frac{n^2 \pi^2}{\lambda^2}, \; n \geq 1.$$

Putting everything together:

$$\psi_n(x) = c_n \sin(\sqrt{\lambda_n} x)$$

and:

$$u_n(x,t) = b_n e^{-k\lambda_n t} \sin(\sqrt{\lambda} x)$$

so the general solution is:

$$u(x,t) = \sum_{n=1}^{m} b_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right) \tag{21}$$

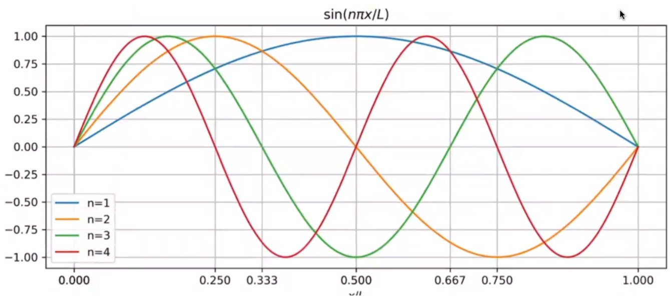
Lastly, to solve for b_n, we use the initial condition:

$$u(x,0) = \sum_{n=1}^{m} b_n \sin\left(\frac{n\pi}{L} x\right) = f(x)$$

Note:

> The above procedure basically invovled:
>
> 1. separate the equations into "eigenfunction+eigenvalue" form:
>
> $$\begin{cases} G'(t) = -k\lambda G(t), & G(t) = ce^{-k\lambda t} \\ \psi''(x) = -\lambda\psi(x), & \psi(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x \end{cases}$$
>
>   so that **the form of solution (eigenfunctions)** is solvable already *unique up to a constant, which can be later determined by initial conditions* .
>
> 2. then, to solve for the **eigenvalue**, we need the **boundary conditions**
>
> $$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$
>
> 3. finally, to solve for the **constants** inside the equation, we needed the **initial conditions**
>
> 4. Then, since it is homogeneous and linear, **any solution can be written as a linear combination of those eigenfunctions**:
>
> $$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi\frac{x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Properties of the above function $u(x,0)$ **include**:



where:

1. the higher the n, the **more the wiggles** in the function
2. the value of $n-1$ indicates the **number of nodes** of the function
3. terms with more wiggles are **damped faster** in time, since you have the term $e^{-k\left(\frac{n\pi}{t}\right)^2 t}$
   - this is because the more spikes => the stronger the gradients => larger heat flows => temperature falls faster

Now, all we need to is to show you **explicitly how to compute the** B_n **using the initial conditions**, using Fourier Series.

2.5 Fourier Series for Heat Equation

Consider an initial condition $f(x)$ for a linear and homogeneous heat equation:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi \frac{x}{L}\right)$$

Note:

> In fact, **not all functions can be written as a Fourier series above.** Sometimes, the sum might not converge.

Heuristics:

1. Instead of dealing with the single equation, consider many individual points $0 \le x_i \le L$, such that we have $\{x_1, x_2, ..., x_k\}$:

$$\begin{cases} f(x_1) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi \frac{x_1}{L}\right) \\ f(x_2) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi \frac{x_2}{L}\right) \\ ... \\ f(x_k) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi \frac{x_k}{L}\right) \end{cases}$$

2. then, we transform the above into matrix form:

$$\vec{F} = \begin{bmatrix} f(x_1) \\ ... \\ f(x_k) \end{bmatrix}, \vec{V_n} = \begin{bmatrix} \sin\left(n\pi \frac{x_1}{L}\right) \\ ... \\ \sin\left(n\pi \frac{x_k}{L}\right) \end{bmatrix}$$

then since:

$$\vec{F} \approx B_1 V_1 + B_2 V_2 + ... + B_k V_k$$

we get:

$$\vec{F} \approx V\vec{B}, \quad V = \left[\vec{V_1}, \vec{V_2}, ..., \vec{V_k}\right]$$

3. Now, **since V might not be square, we use least square to solve for $\vec{B}$**:

$$\begin{aligned} V^T\vec{F} &= V^T V\vec{B} \\ \hat{\vec{B}} &= \left(V^T V\right)^{-1} V^T \vec{F}. \end{aligned}$$

4. suppose $\{\vec{V_1}, \vec{V_2}, ..., \vec{V_n}\}$ are **orthogonal** to each other (**already hints at the notion that integrals of functions are orthogonal**):

$$V^T V = \begin{bmatrix} \vec{V_1}^T \\ ... \\ \vec{V_k}^T \end{bmatrix} \begin{bmatrix} \vec{V_1} & ... & \vec{V_k} \end{bmatrix} = \begin{bmatrix} \vec{V_1}^T\vec{V_1} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \vec{V_k}^T\vec{V_k} \end{bmatrix}$$

then you get:

$$
\begin{aligned}
\hat{\vec{B}} &= \left(V^T V\right)^{-1} V^T \vec{F} \\
&= \begin{bmatrix} \frac{1}{\vec{V}_1^T \vec{V}_1} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \frac{1}{\vec{V}_k^T \vec{V}_k} \end{bmatrix} \begin{bmatrix} \vec{V}_1^T \vec{F} \\ ... \\ \vec{V}_k^T \vec{F} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\vec{V}_1^T \vec{F}}{\vec{V}_1^t \vec{V}_1} \\ ... \\ \frac{\vec{V}_k^T \vec{F}}{\vec{V}_k^T \vec{V}_k} \end{bmatrix}.
\end{aligned}
$$

Now, notice that we solved $\vec{B}$ with essentially $\vec{V}_i$. But what does it mean to have:

$$\vec{V}_n^T \vec{F} = \sum_i \vec{V}_{n_i} \vec{F}_i$$

where $\vec{V}_n = \begin{bmatrix} \sin\left(n\pi \frac{x_1}{L}\right) \\ ... \\ \sin\left(n\pi x_{\frac{k}{L}}\right) \end{bmatrix}$

It means that we are just **summing over the product of the two functions at each point x_i!**

$$
\begin{aligned}
\sum_i^{\infty} \vec{V}_{m_i} \vec{F}_i &= \int_0^L \sin\left(m\pi \frac{x}{L}\right) f(x)\, dx \\
&= \int_0^L \sin\left(m\pi \frac{x}{L}\right) \sum_{n=1}^{\infty} B_n \sin\left(n\pi \frac{x}{L}\right) dx.
\end{aligned}
$$

since the above is just a *sum of integrals*, we just need to consider two cases:

1. $m = n$, then the term:

$$\int_0^L \sin\left(n\pi \frac{x}{L}\right)^2 dx = \frac{L}{2}$$

2. $m \neq n$, then all the terms:

$$\int_0^L \sin\left(m\pi \frac{x}{L}\right) \sin\left(n\pi \frac{x}{L}\right) dx = 0$$

which *hints at orthogonality*

Theorem 2.6: Fourier Series in Heat Equation

Using the conclusions from the above, we get essentially (for a linear and homogeneous heat equation):

$$
\begin{aligned}
\int_0^L \sin\left(n\pi \frac{x}{L}\right) f(x)\, dx &= \int_0^L B_n \sin\left(n\pi \frac{x}{L}\right) dx \\
&= B_n \frac{L}{2}.
\end{aligned}
$$

Hence, we **computes the coefficient** B_n:

$$B_n = \frac{2}{L}\int_0^L \sin\left(n\pi\frac{x}{L}\right) f(x)\,dx \tag{22}$$

Theorem 2.7: Solution of Linear and Homogenous Heat Equation

Combining theorem 2.3 and theorem 2.2, we have, for a linear and homogeneous heat equation:

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L \sin\left(n\pi\frac{x}{L}\right) f(x)\,dx\right)\sin\left(n\pi\frac{x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t} \tag{23}$$

where:

- $f(x)$ is the initial condition of $u(x,0)$

Note:

The equation (23) has an advantage of:

1. you can **truncate the solution series for approximation**
2. If you have a **nice** $f(x)$, then the series (especially the "Fourier term") **would converge**

***Example*: Simple Initial Condition**

Consider the initial condition:

$$u(x,0) = 1$$

then just need to solve B_n and plug that into equation (23):

$$\begin{aligned} B_n &= \int_0^L 1\cdot\sin\left(n\pi\frac{x}{L}\right)dx \\ &= \frac{2}{n\pi}\left(1-\cos\left((n\pi)\right)\right). \end{aligned}$$

hence:

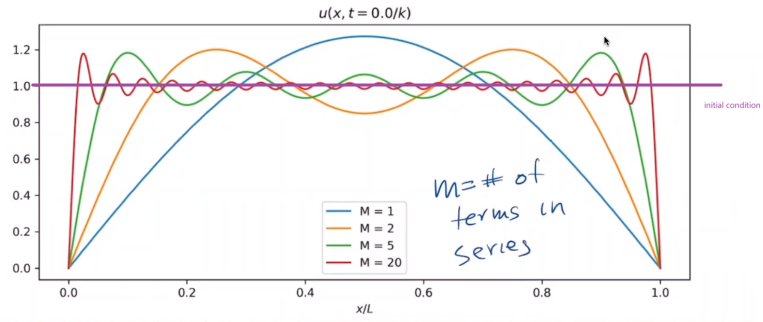
$$B_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

so the overall solution becomes:

$$u(x,t) = \sum_{n=\text{odd}}^{\infty}\frac{4}{n\pi}\sin\left(n\pi\frac{x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

If we approximate the above solution with $f(x) = 1$, they look like:

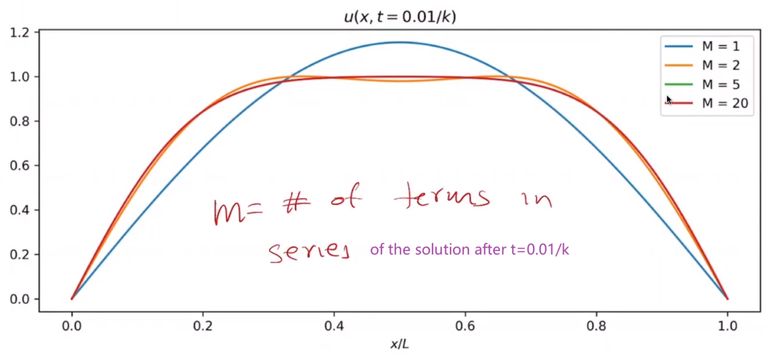
$$u(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{4}{n\pi} \sin\left(n\pi \frac{x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$



where:

- obvious the more the terms, the better the prediction.

and for some time later:



3 Properties of Heat Equation

3.1 Uniqueness of 1D Heat Equation

Theorem 3.1: Uniquesness of 1D Heat Equation

For a 1-D Homogeneous Heat Equation with Homogeneous **Dirichlet** boundary condition:

$$\begin{cases} \text{PDE becomes:} & u_t = ku_{xx} \\ \text{B.C. becomes:} & u(0,t) = u(L,t) = 0 \\ \text{I.C. Is still:} & u(x,0) = f(x) \end{cases}$$

the solution $u(x,t)$ **exists and is unique**

The existence proof is not yet shown here.

Note:

In general, a typical idea for proving the two would be:

- existence: construct a solution from ground up
- uniqueness: *assume that some other solution exists*, and show that it **must be identical** to the original

Proof. The uniqueness of the heat equation can be proven using a **monotonously decreasing quantity** $\int_0^L \frac{w^2}{2}\, dx$:

Suppose that another solution $v(x,t)$ exists, which is not identical to the solution $u(x,t)$. This means we have a non-zero quantity:

$$w(x,t) = u(x,t) - v(x,t) \neq 0$$

Since the PDE itself is homogeneous and linear, and that both u, v are solutions, the linear combination is also a solution:

$$\begin{cases} \text{PDE becomes:} & w_t = kw_{xx} \\ \text{B.C. becomes:} & w(0,t) = w(L,t) = 0 \\ \text{I.C. becomes:} & w(x,0) = f(x) - f(x) = 0 \end{cases}$$

Intuition tells us $w(x,t) = 0$ should be for all time because:

1. there is no source term
2. there is no heat flow at ends
3. initial condition is 0

Consider the integral:

$$\int_0^L w\left(w_t - Kw_{xx}\right)\, dx = 0$$

separating it into two and integrating by parts:

$$\frac{1}{2}\int_0^L \frac{\partial}{\partial t} w^2\,dx = \frac{\partial}{\partial t}\int_0^L \frac{w^2}{2}\,dx$$

$$K\int_0^L ww_{xx}\,dx = ww_x|_0^L - \int_0^L w_x^2\,dx = -\int_0^L w_x^2\,dx$$

So you have:

$$\frac{\partial}{\partial t}\int_0^L \frac{w^2}{2}\,dx = -K\int_0^L w_x^2\,dx \leq 0$$

since $-K < 0$ and integral of a non-negative quantity ≥ 0.

Therefore, we have a monotounsly decreasing quantity, and:

$$\int_0^L \frac{w^2(x,0)}{2}\,dx = 0 \geq \int_0^L \frac{w^2(x,t)}{2}\,dx$$

therefore, it must be that:

$$w(x,t) = 0$$

this contradicts the assumption, hence the solution is unique. □

3.2 Heat Equation with Neumann's Condition

In short, you have the B.C. with derivatives. So the changes are not that big.

Theorem 3.2: Neumann's Condition for 1D Heat Equation

For a 1-D Homogeneous Heat Equation with Homogeneous **Neumann's** boundary condition:

$$\begin{cases} \text{PDE becomes:} & u_t = ku_{xx} \\ \text{B.C. becomes:} & u_x(0,t) = u_x(L,t) = 0 \\ \text{I.C. Is still:} & u(x,0) = f(x) \end{cases}$$

the solutions takes the form:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(n\pi \frac{x}{L}\right) e^{-K\left(n\frac{\pi}{L}\right)^2 t} \tag{24}$$

with the coefficients computed from initial condition:

$$A_n = \begin{cases} \frac{2}{L}\int_0^L \cos\left(n\pi\frac{x}{L}\right) f(x)\,dx, & n > 0 \\ \frac{1}{L}\int_0^L f(x)\,dx, & n = 0 \end{cases}$$

Reminder:

To solve seperable equations, we usually do the following:

1. *separate the PD into eigenfunction form using the homogeneous equations*
2. *solve eigenfunction based on the possible eigenvalues*
3. *solve the coefficients of the eigenfunctions based on the* ***inhomogeneous*** *initial condition*

Proof. Using the above general procedure:

1. let $u(x,t) = \phi(x)G(t)$, and we have:

$$\begin{cases} G(t) = Ae^{-K\lambda t} \\ \phi(x)'' = -\lambda\phi \end{cases}$$

2. using the boundary condition, we solve for λ:
   - if $\lambda > 0$, we have:

$$\phi(x) = c_1 \sin\left(\sqrt{\lambda}x\right) + c_2 \cos\left(\sqrt{\lambda}x\right)$$

plugging in the boundary condition, we get:

$$c_1 = 0, \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

hence the non-trivial solution is:

$$\phi_n(x) = c_2 \cos\left(n\pi \frac{x}{L}\right), \; n = 1, 2, 3...$$

- if $\lambda = 0$, we have:

$$\phi(x) = c_1 + c_2 x$$

plugin in the boundary condition:

$$c_2 = 0$$

so we get a **non-trivial solution:**

$$\phi(x) = c_1 = A_0$$

- if $\lambda < 0$, we will have only the possibility of a trivial solution, namely $\phi(x) = 0$.

hence, the overall solution for eigenvalue and eigenfunction pairs are the **sum of all possibilities:**

$$\phi_n(x) = c_n \cos\left(n\pi \frac{x}{L}\right), \; \lambda_n = \left(\frac{n\pi}{L}\right)^2, \; n = 0, 1, 2, 3...$$

3. Lastly, we use superposition of eigenfunctions and the initial condition to filter out some of them:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(n\pi \frac{x}{L}\right) e^{-K\left(\frac{n\pi}{L}\right)^2 t}, \; n = 0, 1, 2...$$

but note that the **orthogonality condition is different:**

$$\int_0^L \cos\left(m\pi \frac{x}{L}\right) \cos\left(n\pi \frac{x}{L}\right) dx = \begin{cases} 0, & m = n \\ \frac{L}{2}, & m = n \neq 0 \\ L, & m = n = 0 \end{cases}$$

therefore, the coefficients are calculated to be:

$$A_n = \begin{cases} \frac{2}{L} \int_0^L \cos\left(n\pi \frac{x}{L}\right) f(x)\, dx, & n > 0 \\ \frac{1}{L} \int_0^L f(x)\, dx, & n = 0 \end{cases}$$

□

Note:

> Solution of the form of equation (34) also hints at the **stable state solution** of $u(x,t) \to A_0 =$ average of initial thermal energy, since basically we have:
>
> $$A_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

3.3 Heat Equation in Multi dimension

Reminder:

> *For multi dimension **homogenous** heat equation, we have:*
>
> $$u_t = K\nabla^2 u$$

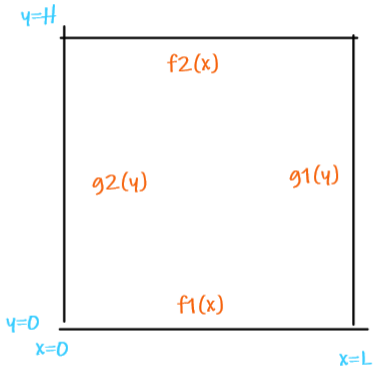
3.3.1 Square Box Steady State Solution

The steady state solution becomes:

$$\nabla^2 u = 0$$

which is also called **the Laplace's equation.**

Since we are in 2-D, consider a rectangle:



so our system is:

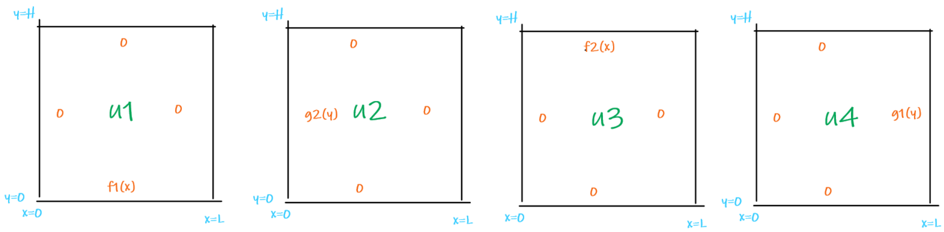
$$\begin{cases} \text{PDE becomes:} & u_{xx} + u_{yy} = 0 \\ \text{B.C. becomes:} & \begin{cases} u(x,0) = f_1(x) \\ u(x,H) = f_2(x) \\ u(L,y) = g_1(y) \\ u(0,y) = g_2(y) \end{cases} \\ \text{There is no I.C.} & \end{cases}$$

Note:

> Here, we see that we are having **inhomogeneous boundary conditions, which makes the calculation cumbersome**.
>
> - to solve this issue, we use the **superposition property**

1. Consider splitting the solution into four pieces:

$$u = u_1 + u_2 + u_3 + u_4$$



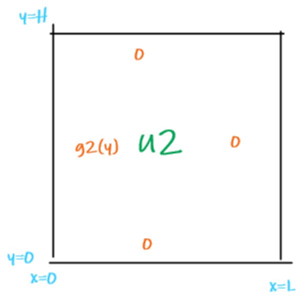
this would work because:

- $\nabla^2 u = 0$ is homogenous
- the boundary condition adds up correctly

this means, for each equation, you have:

- **3 homogeneous B.C.**
- **1 inhomogenous B.C.**

Consider the solution u_2:



Let $u_2 = h(x)\varphi(y)$. Then, **separating the variables**:

$$\nabla^2 u_2 = 0$$
$$\frac{h''}{h} = -\frac{\varphi''}{\varphi} \equiv \lambda.$$

2. Now, **calculating the eigenfunctions from *homogenous* boundary conditions**:

$$\begin{cases} \varphi(0) = 0 \\ \varphi(H) = 0 \\ h(0)\varphi(y) = g_2(y) \quad \text{inhomogenous} \\ h(L) = 0 \end{cases}$$

first calculating φ:

$$\varphi'' = -\lambda\varphi, \quad \begin{cases} \varphi(0) = 0 \\ \varphi(L) = 0 \end{cases}$$

the solution is simply:

$$\varphi_n(y) = A_n \sin\left(n\pi \frac{y}{H}\right), \quad \lambda_n = \left(n\frac{\pi}{H}\right)^2$$

now, for h, we have:

$$h'' = \left(\frac{n\pi}{H}\right)^2 h, \quad \left\{h(L) = 0\right.$$

so the general solution is:

$$h(x) = c_1 \cosh\left(n\pi \frac{x}{H}\right) + c_2 \sinh\left(n\pi \frac{x}{H}\right)$$

to make the *boundary condition work better*, we use the fact that the below also works as a solution:

$$h(x) = c_1 \cosh\left(\frac{n\pi}{H}(x - L)\right) + c_2 \sinh\left(\frac{n\pi}{H}(x - L)\right)$$

hence with the only homogeneous boundary condition, we have:

$$h_n(x) = c_2 \sinh\left(\frac{n\pi}{H}(x - L)\right)$$

3. Last, we **combine them and solve the coefficient using the *inhomogenous boundary condition***

$$u_2(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(n\pi \frac{y}{H}\right) \sinh\left(n\frac{\pi}{H}(x - L)\right)$$

and that:

$$u_2(0, y) = g_2(y) = -\sum_{n=1}^{\infty} A_n \sin\left(n\pi \frac{y}{H}\right) \sinh\left(n\pi \frac{L}{H}\right)$$

using the orthogonality, we get:

$$A_n = -\frac{2}{H} \frac{1}{\sinh\left(n\pi \frac{L}{H}\right)} \int_0^H g_2(y) \sin\left(n\pi \frac{y}{H}\right) dy$$

4. Then, you need to compute the rest of u_1, u_3, u_4 using a similar procedure.

Theorem 3.3: Parital Solution for Equilibrium Heat Equation in Square

For a 2-D heat equation inside a square box, the **steady state solution**

has:

$$\begin{cases} \text{PDE becomes:} & u_{xx} + u_{yy} = 0 \\ \text{B.C. becomes:} & \begin{cases} u(x,0) = f_1(x) \\ u(x,H) = f_2(x) \\ u(L,y) = g_1(y) \\ u(0,y) = g_2(y) \end{cases} \\ \text{There is no I.C.} & \end{cases}$$

and for $u = u_1 + u_2 + u_3 + u_4$, one of them takes the form:

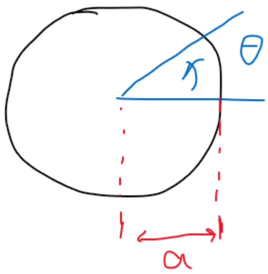
$$u_2(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(n\pi \frac{y}{H}\right) \sinh\left(n\frac{\pi}{H}(x-L)\right) \tag{25}$$

with:

$$A_n = -\frac{2}{H} \frac{1}{\sinh\left(n\pi \frac{L}{H}\right)} \int_0^H g_2(y) \sin\left(n\pi \frac{y}{H}\right) dy \tag{26}$$

3.3.2 Circular Disk Steady State Solution

Now, consider the following setup:



with boundary condition:

$$B.C. = u(a,\theta) = f(\theta)$$

in the *domain* of:

$$0 \le r \le a, \quad -\pi \le \theta \le \pi$$

then, again the idea is always the same.

1. **Separate the variable** of the PDE (Laplace's Equation since we are at a steady state)

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Let $u(r,\theta) = G(r)\varphi(\theta)$:

$$\frac{1}{r}[r(G\varphi)_r]_r + \frac{1}{r^2}(G\varphi)_{\theta\theta} = 0$$

$$\frac{r}{G}[rG']_r = -\frac{\varphi''}{\varphi} = \lambda.$$

Note:

> Positive λ instead of $-\lambda$ is chosen here, because of the linear algebra analogy:
>
> $$Ax = \lambda x$$
>
> has λ being positive if A is *positive definite*. In this case, it will be proven later that the operator $-\frac{\partial^2}{\partial\theta^2}$ is positive definite.

2. Then, we need to use **boundary conditions to solve for eigenfunctions**. However, in this question, some *hidden homogeneous boundary conditions* are:

$$\begin{cases} |u(0,\theta)| < \infty, & \text{polar coordinates are singular at } r = 0 \\ \varphi(-\pi) = \varphi(\pi), & \text{continuity} \\ \varphi'(-\pi) = \varphi'(\pi) \end{cases}$$

Note:

> They are homogeneous conditions because: **linear combination of solutions satisfying those conditions will still satisfy those conditions**

Now, solving for φ, for $\lambda > 0$:

$$\varphi(\theta) = c_1 \sin\left(\sqrt{\lambda}\theta\right) + c_2 \cos\left(\sqrt{\lambda}\theta\right)$$

using the boundary conditions:

$$\varphi(\pi) = \varphi(-\pi)$$
$$2c_1 \sin\left(\sqrt{\lambda}\pi\right) = 0.$$

similarly, the other condition:

$$\varphi'(\pi) = \varphi'(-\pi)$$
$$2c_2\sqrt{\lambda} \sin\left(\sqrt{\lambda}\pi\right) = 0.$$

now, if $\sin\left(\sqrt{\lambda}\pi\right) \neq 0$, *then we have $c_1 = c_2 = 0$, which becomes a trivial solution*. Therefore, it means $\sin\left(\sqrt{\lambda}\pi\right) = 0$:

$$\lambda_n = n^2, \quad n = 1, 2, 3...$$

and **$\varphi(\theta)$ is still a linear combination of two independent solutions**:

$$\varphi_n(\theta) = c_1 \sin(n\theta) + c_2 \cos(n\theta)$$

for $\lambda = 0$, we have the linear solution:

$$\varphi = c_3\theta + c_4$$

using the same boundary conditions, we get:

$$\varphi_0(\theta) = c_4$$

(no solution if $\lambda < 0$).

Hence, combining we get:

$$\varphi_n(\theta) = c_1 \sin(n\theta) + c_2 \cos(n\theta), \quad n = 0, 1, 2, ..$$

Now, we do the same to find out G:

$$\begin{aligned} \frac{r}{G}(rG')' &= \lambda = n^2 \\ r^2G'' + rG' - n^2G &= 0. \end{aligned}$$

which hints at $G = r^p$ for some power p. Plugging the solution in:

$$\begin{aligned} r^2 p(p-1)r^{p-2} + rpr^{p-1} - n^2 r^p &= 0 \\ p &= \pm n. \end{aligned}$$

Hence, we get:

$$G_n(r) = d_1 r^n + d_1 r^{-n}, \quad n = 0, 1, 2...$$

but for $n = 0$, we also need two solutions, since **we are in 2-D, there should be two linearly independent solutions**.

Going back at the PDE, when $n = 0$:

$$\begin{aligned} (rG')' &= 0 \\ G' &= \frac{d_3}{r} \\ G &= d_3 \ln(r) + d_4. \end{aligned}$$

Now we have **correctly:**

$$G_n(r) = \begin{cases} d_1 r^n + d_2 r^{-n}, & n = 1, 2, 3... \\ d_3 \ln(r) + d_4, & n = 0 \end{cases}$$

additionlly, we also had the homogenous condition:

$$|u(r, \theta)| = |G(0)| < \infty$$

this *simplifies some eigenfuunctions*:

$$G(r) = d_n r^n, \quad n = 0, 1, 2, 3...$$

3. Now, we have used up all the **three homogeneous boundary condition, we use the remaining *inhomogenous* one to figure out the coefficients**:

$$u(r, \theta) = \sum_{n=0}^{\infty} (A_n \cos(n\theta) + B_n \sin(n\theta)) r^n$$

using the inhomogenous condition:

$$f(\theta) = \sum_{n=0}^{\infty} \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) a^n.$$

for A_n, we just need to use $\cos(n\theta)$, and for B_n, we use $\sin(n\theta)$:

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta \\ n \geq 1, \quad A_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta \\ B_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta. \end{aligned}$$

Note:

> Notice that A_0 is the *average of the values on the boundary of a circle*. More importantly, notice that the **center of the circle has the value of the average of the boundary**:
>
> $$u(0,\theta) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$
>
> where for our setup, $r = 0$ is the center.

3.4 Properties of Laplace's Equation

From subsubsection 3.3.2, we see the mean value property.

Theorem 3.4: Mean Value Property

In general, any solution that satisfies a Laplace equation $\nabla^2 u = 0$ has the property of: **the value at the center of any circle is the average of the values on the boundary of the circle**.

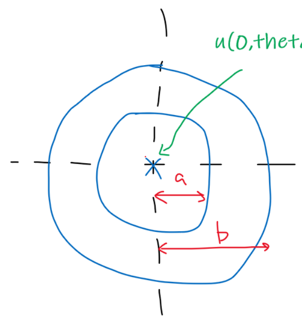
$$u(x_0, y_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + r_0 \cos(\theta), y_0 + r_0 \sin(\theta))\, d\theta \tag{27}$$

for the previous case of a circular disk, we see if *evidently:*

$$u(0,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a,\theta)\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

functions that satisfy this property is also called a **harmonic function**.

An example of the above would be:



So that:

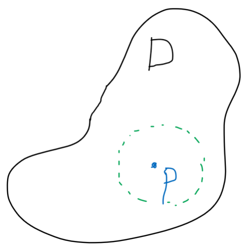
$$u(0,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a,\theta)\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(b,\theta)\, d\theta$$

if inside both circle, the equation $\nabla^2 u = 0$ *is satisfied*.

Theorem 3.5: Max and Min Principle

For any u inside a domain D, such that $\nabla^2 u = 0$, the **maximum and the minimum of u can only occur at the boundary**.

Proof. Consider the following setup:



we have inside the domain D:

- u is not a constant
- p is a point inside the domain D

Suppose towards contradiction that p is the maximum. Then we can draw a circle around p, such that:

$$Max = u(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\text{on the circle's boundary})\, d\theta$$

However, if $u(p)$ is max, then $u(\text{on the boundary}) < u(p)$, which is a contradiction since **average of a smaller quantity cannot become** $u(p)$.

Therefore, p cannot be within the region D. The same arguments go for a minimum of u. In the end, we conclude that both minimum and maximum must occurs at the *boundary of u*. ☐

Note:

> The min and max property for a solution that satisfies Laplace's equation makes sense because:
>
> - if min is inside the region, then *heat will flow towards that point in the region*, and hence we do not have a steady state any more
> - if max is inside the region, then *heat will flow out from that point in the region*, and hence we do not have a steady state any more

Reminder:

> *The Laplace equation comes from the heat equation where:*
>
> $$u_t = 0 = K\nabla^2 u$$
>
> *which signifies a steady state solution.*

3.4.1 Uniqueness of Laplace Equation

Theorem 3.6: Uniquess with Dirichlet B.C.

Solutions of Laplace's equation is **unique** if we have a **Dirichlet's boundary condition**:

$$\nabla^2 u = 0$$
$$u|_{\partial D} = f.$$

for u inside a domain D

Proof. Using the *same trick*, suppose there is another solution v that works as well. Then, computing the quantity $w = u - v$:

$$\nabla^2 w = 0$$
$$w|_{\partial D} = 0.$$

due to the *linearity of* ∇^2.

Then, by the **min and max principle**, the solution is bounded by **min and max that is also on the boundary**. Therefore:

$$0 \leq w \leq 0$$

hence, we conclude that $w = 0$ and the solution is unique. □

Note:

In fact, Laplace's equation with **Neumann's Condition will not be unique** (depends on initial condition). This is because if only specifies:

$$\nabla^2 u = 0$$
$$\frac{\partial u}{\partial \vec{n}} = 0.$$

so that I can just have $v = u + k$, for k being an arbitrary constant and it will still work.

3.4.2 Solubility/Compatibility of Laplace's Equation

Theorem 3.7: Solubility/Compatibility

Laplace's equation **has a solution only when the following condition is satisfied:**

$$\oint \vec{\nabla} u \cdot \hat{n}\, dS = 0$$

which means that the **net heat flow into/out of the region is** 0 (very sensible).

Proof. For Laplace's equation to work, we need a u that satisfies:

$$\nabla^2 u = 0$$

this means that if we consider the region where u lives:

$$\begin{aligned}\int_D \nabla^2 u \, dV &= \int_D \vec{\nabla} \cdot \left(\vec{\nabla} u\right) \, dV \\ &= \oint_{\partial D} \vec{\nabla} u \cdot \hat{n} \, dS \\ &= 0.\end{aligned}$$

since $\nabla^2 u = 0$.

Therefore, it means

$$\oint \vec{\nabla} u \cdot \hat{n} \, dS = 0$$

where $\vec{\nabla} u \cdot \hat{n}$ signifies the net heat flow into/out of the region D. □

3.4.3 Continuous Variability of Laplace Equation

Theorem 3.8: Continuous Variability of Laplace's Equation

For a Laplace equation with **Dirichlet's boundary condition**, the **solution varies continuously on the "data"** (e.g. Initial condition, boundary condition, etc), such that *uncertainties of the initial data corresponds to uncertainty of the solution*:

$$\begin{aligned}&\nabla^2 u = 0 \\ &u|_{\partial D} = f(\vec{x}) + uncert(\vec{x}).\end{aligned}$$

then:

$$\min_{\vec{x} \in \partial D} (uncert(\vec{x})) \leq uncert(u) \leq \max_{\vec{x} \in \partial D} (uncert(\vec{x}))$$

Proof. This again comes from the *min and max principle*.

Consider two setups:

$$\begin{cases} \nabla^2 u = 0, & u|_{\partial D} = f(\vec{x}) \\ \nabla^2 v = 0, & v|_{\partial D} = g(\vec{x}) \end{cases}$$

then for the function $w = u - v$, I have:

$$\begin{aligned}\nabla^2 w &= 0 \\ w &= g(\vec{x}) - f(\vec{x}).\end{aligned}$$

hence

$$\min_{\vec{x} \in \partial D} (g(\vec{x}) - f(\vec{x})) \leq w \leq \max_{\vec{x} \in \partial D} (g(\vec{x}) - f(\vec{x}))$$

if $g(\vec{x}) - f(\vec{x})$ is small, then the uncertainty w is also small. □

Note:

> The *take away message is that, the Dirichlet's boundary condition controls the behavior of u inside the domain*.

3.4.4 Incompressibility and Laminar Flow

Recall that the *conservation of mass equation* (from subsection 2.1)

$$\rho_t + (\rho u)_x = Q$$

In 3-D, for an incompressible fluid with no source, we have:

$$\rho_t + \rho \vec{\nabla} \cdot \vec{u} = 0$$

where $\vec{u}$ denotes velocity.

Theorem 3.9: Incompressibility

If the fluid satisfies the Laplace's equation, and that ρ is constant, this means:

$$\vec{\nabla} \cdot \vec{u} = 0$$

which means solution will be in the shape of the **level curve of $\vec{\psi}$**. So that in 2-D, we have:

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \psi_y \\ -\psi_x \end{bmatrix}$$

and *$\psi(x, y)$ is called the stream function*.

Proof. Let the 2-D u be in the form:

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and we need if ρ is a constant (incompressible):

$$u_{1_x} + u_{2_y} = 0$$

substituting in $u = \begin{bmatrix} \psi_y \\ -\psi_x \end{bmatrix}$, we see that it gives us $\vec{\nabla} \cdot \vec{u} = 0$. *Additionally:*

$$\nabla\psi \cdot u = \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix} \begin{bmatrix} \psi_y \\ -\psi_x \end{bmatrix} = 0$$

This means that u flows in the direction of ψ:

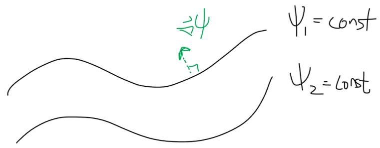
□

Theorem 3.10: Irrotationability/Laminar Flow

A fluid is irrotational if (from the same mass conservation equation):

$$\vec{\nabla} \times \vec{u} = 0$$

this means that, if the fluid is **also incompressible**, then we have a



stream function as the solution, and that:

$$\vec{\nabla} \times \begin{bmatrix} \psi_y \\ -\psi_x \end{bmatrix} = -\psi_{xx} - \psi_{yy} = \nabla^2 \psi = 0$$

which is the **Laplace's Equation again!**

4 Fourier Series

4.1 Fourier Series Central Theorem

Now, we need to deal with the case that our expressions/solutions of heat equation looks like:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \cos\left(n\pi \frac{x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\pi \frac{x}{L}\right)$$

but we don't even know *if it converges to a function!*

This introduces the Fourier Series and its related properties.

Theorem 4.1: Fourier Series

A Fourier Series **of a function** f **between the interval** $-L \leq x \leq L$ is defined to be:

$$\text{Fourier Series} = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\pi \frac{x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\pi \frac{x}{L}\right) \tag{28}$$

where:

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dL \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(n\pi \frac{x}{L}\right) dx \equiv a_n[f(x)] \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(n\pi \frac{x}{L}\right) dx \equiv b_n[f(x)]. \end{aligned} \tag{29}$$

However, sometimes the **series does not converge**, or if it does, it **may not be equal to** $f(x)$ **at all points.** Therefore, we often use the following notation:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\pi \frac{x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\pi \frac{x}{L}\right)$$

This introduces the following definitions to know.

Definition 4.1: Piecewise Smooth Function

A function $f(x)$ is piecewise smooth if it is *defined, continuous, and continuously differentiable* **except at a finite number of points**, where at those exceptional point there is a **jump discontinuity**.

- this means that at those exceptional points, $f(x_k), f'(x_k)$ may be undefined (*left and right limits be not equal*), but **their left and right limit must exists**

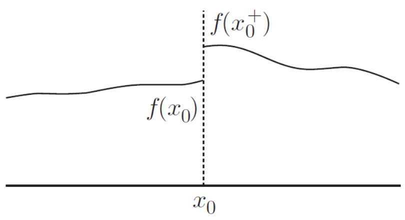
$$f(x_k^-) = \lim_{x \to x_k^-} f(x), \quad f(x_k^+) = \lim_{x \to x_k^+} f(x)$$
$$f'(x_k^-) = \lim_{x \to x_k^-} f'(x), \quad f'(x_k^+) = \lim_{x \to x_k^+} f'(x)$$

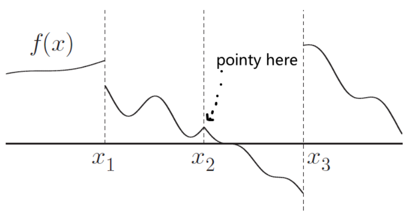
- then obviously for points at the edge, e.g at $x = L, -L$, we only need one side to exist.

In other words, only **finite number of jump discontinuities for $f(x_k)$ and $f'(x_k)$** are allowed.

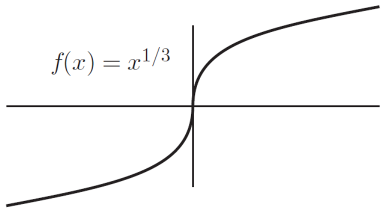
For example:

The following function is piecewise smooth:





The following function is *not* piecewise smooth:



because the left and right limit of $\lim_{x\to 0} f'(x)$ *does not exist, since they are* $\to \infty$*, i.e. unbounded*

Definition 4.2: 2L Periodic Extension

A $2L$ periodic extension applied to a function $f(x)$ that is defined within $-L \leq x \leq L$ means **copying and pasting** the function $f(x)$ onto other intervals of $2L$ length, such that the resulting function $\tilde{f}(x)$ is *defined for all x*.

For Example:

Consider the function $f(x) = \frac{3}{2}x$ for $-L \leq x \leq L$. The 2L periodic extension of it means you have:

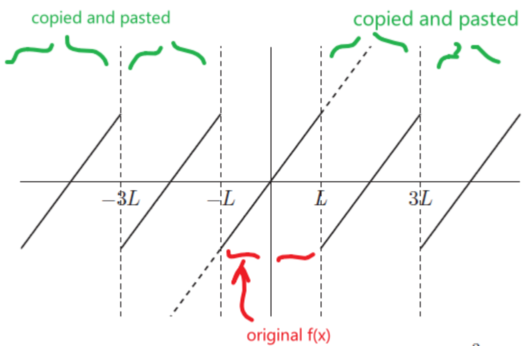


FIGURE 3.1.4 Periodic extension of $f(x) = \frac{3}{2}x$.

where the period of the copied function is $2L$.

Note:

Notice that the *copied function may not be continuous* for the new domain of all x.

Finally, to the convergence:

Theorem 4.2: Convergence Theorem on Fourier Series

If $f(x)$ is **piecewise smooth** on the interval $-L \leq x \leq L$, then the Fourier Series of $f(x)$ **converges to**:

1. the **periodic extension of $f(x)$**, where the periodic extension is **continuous**

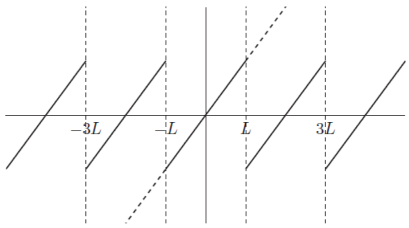
2. the **average of the left and right limits of the periodic extension**,
$$\frac{1}{2}[f(x_+) + f(x_-)]$$
where the periodic extension **has a jump discontinuity**

For Example:
Consider a function $f(x) = \frac{3}{2}x$ defined on the interval $|x| \leq L$ we had before. Since $f(x)$ is piecewise continuous, then Fourier Series of it:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\pi\frac{x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\pi\frac{x}{L}\right)$$

converges to :

1. the following for points where the graph is continuous



2. converges to $\frac{1}{2}[\tilde{f}((2m-1)\,L) + \tilde{f}((2m+1)\,L)] = 0$ for the discontinuous points.

   - notice that we get an *infinite number of discontinuities*, which is allowed since we allowed $f(x)$ to have a finite number of jump discontinuity, and hence the 2L periodic extension would of course have infinite repeated number of it.

In this case, since $f(x)$ is an *odd function, we can simplify the Fourier Series to*:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(n\pi\frac{x}{L}\right)$$

Corollary 4.1: Fourier Series with Odd Functions

For any *piece-wise smooth* **odd function** $f(x)$, the Fourier Series converges, and the *series itself can be simplified to*

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(n\pi \frac{x}{L}\right)$$

because:

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\, dL = 0 \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\pi\frac{x}{L}\right) dx = 0 \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\pi\frac{x}{L}\right) dx \equiv \frac{2}{L}\int_{0}^{L} f(x)\sin\left(n\pi\frac{x}{L}\right) dx.
\end{aligned}$$

Corollary 4.2: Fourier Series with Even Functions

Similar to the above, *for any piece-wise smooth* **even function** $f(x)$, the Fourier Series converges, and the *series can be simplified to:*

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \sin\left(n\pi \frac{x}{L}\right)$$

because:

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\, dL = \frac{1}{L}\int_{0}^{L} f(x) dL \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\pi\frac{x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(n\pi\frac{x}{L}\right) dx \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\pi\frac{x}{L}\right) dx = 0.
\end{aligned}$$

Note:

This means the **general procedure for computing the convergence of Fourier Series** of a function $f(x)$ looks like:

1. check if $f(x)$ is *piece-wise smooth* in the interval $-L \leq x \leq L$
2. Looks at the 2L periodic extension of $f(x)$, which is $\tilde{f}(x)$, and compute the *discontinuous points*
3. Lastly, *compute the coefficients* a_0, a_n, b_n

4.2 Using Fourier Series for Heat Equation

Recall that for heat equations, we had solutions *in the domain of* $0 \leq x \leq L$. However, the nice theorems of Fourier Series required the domain **from** $-L \leq x \leq L$. Therefore, we need to *do some tricks*.

1. *extend the initial condition* $f(x)$ *to* $-L \leq x \leq L$, while **keeping the solution the same**(i.e. we want the **coefficients** of the series to be the same)
2. since solutions is now extended, we can **use theorems from Fourier Series**

Example

Consider the heat equation with the following setup:

$$u_t = k u_{xx}$$

with B.C. and I.C.:

$$\begin{cases} u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) = 1 \end{cases}$$

and we know that solutions will look like:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi \frac{x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

and we care about the *convergence of*:

$$\sum_{n=1}^{\infty} B_n \sin\left(n\pi \frac{x}{L}\right)$$

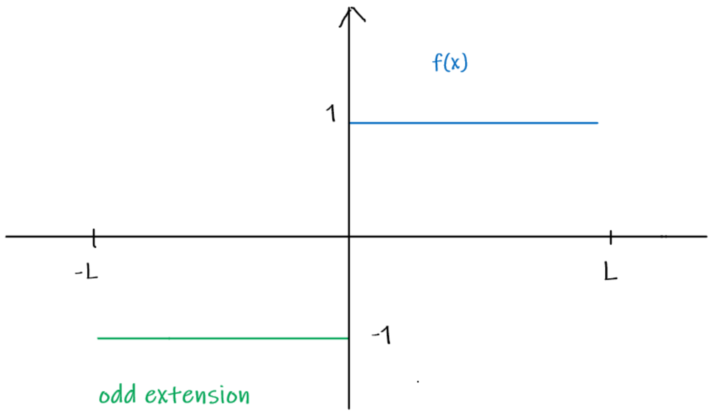
Therefore, to use Theorems from Fourier Series, we need to:

1. *extend the initial condition* so that the Fourier Series of the **extended initial condition** would **match the solution of our PDE**

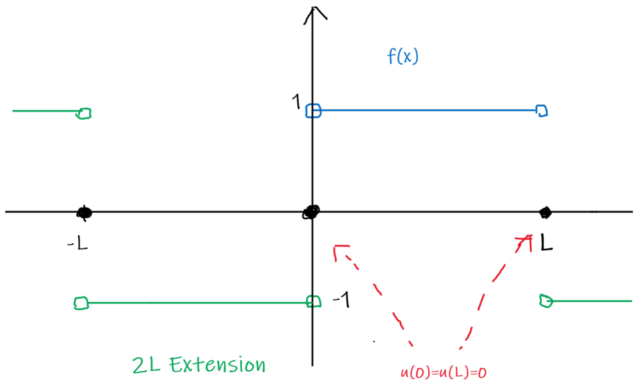
   In this case , this means that we choose the **odd extension of the I.C.** $f(x)$, so that its Fourier Series will only have $\sin(...)$ terms, and so is our PD solution.
2. then, we *use the Fourier Theorem* to see if the solution is convergent and correct

Graphically, we did:

1. step 1:



2. step 2:
First notice that the above is *piece-wise smooth*, therefore the Fourier Theorem 4.2 says that it *converges* to:



where we see that the **Dirichlet's Boundary Condition is satisfied**. This means that the *our solution of infinite sum of* $\sin(\ldots)$ *converges and solves correctly.*

Note:

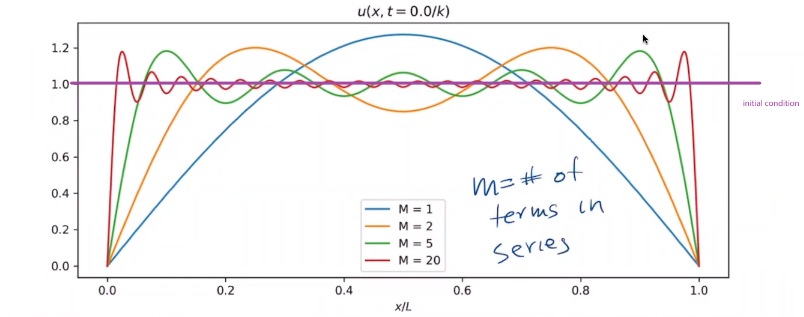
However, for *truncated series/approximation* of the complete Fourier Series, its "accuracy" depends on whether the initial condition is continuous.

If the initial condition $f(x)$ has jump discontinuity, then we have the **Gibb's Phenomena**, which says that the:

- the truncated series is *not uniform* (i.e. accuracy depends on position x, instead of on number of terms)

- there will be an *overshoot of about* 18% of the jump/discontinuity from the I.C.

For instance, the above example with $u(x,0) = 1$ has a jump from 0 to 1, therefore its truncated series has an *overshoot near the discontinuity* and is not uniform:



4.3 Derivatives of Fourier Series

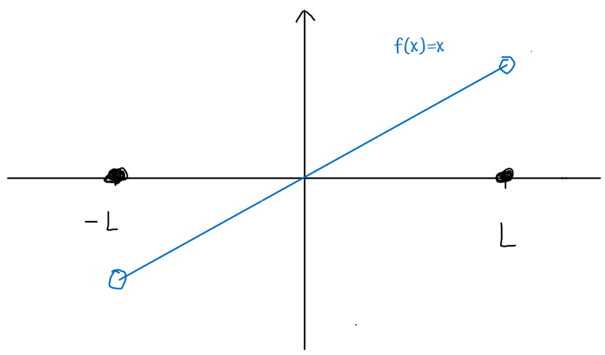
The take away message from this section is:

- not all Fourier Series can be *differentiated* **directly**

Consider the example of $f(x) = x$, for $-L \leq x \leq L$. Since its odd, the Fourier Series look like:

$$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} L}{n\pi} \sin\left(n\pi \frac{x}{L}\right)$$

and its convergence *according to the theorem* will be:



However, computing the derivative **directly gives**:

$$2\sum_{n=1}^{\infty} (-1)^{n+1} \cos\left(n\pi \frac{x}{L}\right) \neq 1$$

since the Fourier Series for 1 should just have $a_0 = 1$ and $a_n = 0$. Therefore, in this case, **we cannot take the derivative directly**.

Theorem 4.3: Differentiating Fourier Series

Suppose we have a Fourier Series for $f(x)$:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\pi \frac{x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\pi \frac{x}{L}\right)$$

the derivative of $f'(x)$ is equal to the derivative of the above Fourier Series (i.e. you can **directly take the derivative**) if and only if:

1. $f'(x)$ is **piecewise-smooth** (i.e. so that the Fourier Series converges)
2. the **2L extension** of $f(x)$ is **continuous**

then, you will have:

$$\begin{aligned} f'(x) &\sim \sum_{n=1}^{\infty} -a_n[f]\left(\frac{n\pi}{L}\right)\sin\left(n\pi\frac{x}{L}\right) + \sum_{n=1}^{\infty} b_n[f]\left(\frac{n\pi}{L}\right)\cos\left(n\pi\frac{x}{L}\right) \\ &= \sum_{n=1}^{\infty} b_n[f']\sin\left(n\pi\frac{x}{L}\right) + \sum_{n=1}^{\infty} a_n[f']\cos\left(n\pi\frac{x}{L}\right). \end{aligned}$$

Proof. If the 2L extension of $f(x)$ is continous, then this means:

- $f(x)$ is continous
- $f(L) = f(-L)$

then consider *computing explicitly:*

$$f'(x) \sim \sum_{n=1}^{\infty} b_n[f']\sin\left(n\pi\frac{x}{L}\right) + \sum_{n=1}^{\infty} a_n[f']\cos\left(n\pi\frac{x}{L}\right).$$

then it means:

$$a_0[f'] = \frac{1}{2L}\int_{-L}^{L} f'(x)\,dx = \frac{1}{2L}\left(f(L) - f(-L)\right) = 0$$

and then:

$$\begin{aligned} a_n[f'] &= \frac{1}{L}\int_{-L}^{L} f'(x)\cos\left(n\pi\frac{x}{L}\right)dx \\ &= \frac{1}{L}\left(f(x)\cos\left(n\pi\frac{x}{L}\right)\Big|_{-L}^{L} + \int_{-L}^{L} f(x)\frac{n\pi}{L}\sin\left(n\pi\frac{x}{L}\right)dx \right) \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\left(\frac{n\pi}{L}\right)\sin\left(n\pi\frac{x}{L}\right)dx \\ &= b_n[f]\left(\frac{n\pi}{L}\right). \end{aligned}$$

as we had above. Similarly:

$$
\begin{aligned}
b_n[f'] &= \frac{1}{L}\int_{-L}^{L} f'(x)\sin\left(n\pi\frac{x}{L}\right)dx \\
&= \frac{1}{L}\left(f(x)\,\sin\left(n\pi\frac{x}{L}\right)\Big|_{-L}^{L} - \int_{-L}^{L} f(x)\frac{n\pi}{L}\cos\left(n\pi\frac{x}{L}\right)dx\right) \\
&= \frac{-1}{L}\int_{-L}^{L} f(x)\left(\frac{n\pi}{L}\right)\cos\left(n\pi\frac{x}{L}\right)dx \\
&= -a_n[f]\left(\frac{n\pi}{L}\right).
\end{aligned}
$$

this completes the proof. □

Note:

The crucial part to make **directly differentiating correct** is to have the following evalauted to 0 in the above proof:

$$
\begin{cases}
f(L) - f(-L) = 0 \\
f(x)\,\cos\left(n\pi\frac{x}{L}\right)\Big|_{-L}^{L} = 0 \\
f(x)\,\sin\left(n\pi\frac{x}{L}\right)\Big|_{-L}^{L} = 0
\end{cases}
$$

Otherwise, you **need to add those extra terms back** when you are doing the **derivatives of the Fourier Series**.

***Example*: Fixing the $f(x) = x$ case**

Therefore, since the 2L extension of $f(x) = x$ is *not continuous*, we need to add the extra terms back when doing the derivatives:

$$
a_0[f'] = \frac{1}{2L}\left(f(L) - f(-L)\right) = 1
$$

which *fixed the a_0 term*, and adding the other terms back, you will get $a_n = 0$.

Corollary 4.3: Derivatvies for Heat Equation

Now, applying to the heat equation, which is *also a Fourier Series* but for $0 \le x \le L$:

1. if the series is cosine, then you can **always take the derivatives of the series directly**, since:
   - $\cos\left(n\pi\frac{x}{L}\right)$ is even, so $f(L) = f(-L)$ (i.e. the *extended initial condition is even*)
   - derivative is smooth
2. if the series is sine, then you can only **take the derivative di-**

rectly when you have Dirichlet's Boundary Condition, since:

- odd function/initial condition/Fourier Series gives $-f(x) = f(-x)$, but we *need* $f(L) = f(-L)$. Therefore, this **forces** $f(L) = f(-L) = 0$, which is the Dirichlet's Boundary Condition.

3. otherwise, you need to **add the extra terms during computation**

4.3.1 Using Fourier Derivatives for Heat Equation

Consider the heat equation with the following setup:

$$u_t = ku_{xx}$$

with B.C. and I.C.:

$$\begin{cases} u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Since we *know that the eigenfunctions for this is* $\sin\left(n\pi\frac{x}{L}\right)$, I now employ the method of **eigenfunction expansion:**

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin\left(n\pi\frac{x}{L}\right)$$

where I used the $=$ sign instead of $\sim$ since the above would be exact since there is *no discontinuities/jumps*.

Now, I want to:

1. plug this back into the PDE by *computing the derivatives*
2. plugging the derivatives back into the PDE and show that we get back the *original solution we had in the previous chapter*

Step 1:

First I need to compute the derivatives of u and u_x, by *assuming that* u, u_t, u_x, u_{xx} *are continuous.* Then, I consider if I can take term-by-term derivatives:

1. odd extension of $u(x,t)$ so that we get the domain $-L \leq x \leq L$
2. the *odd extension is continuous* since we have the boundary condition $u(0,t) = 0$
3. the *2L extension* of the Fourier Series of $u(x,t)$ is also continuous since $u(L,t) = 0$
4. the function u_x is *assumed to be continuous*

Therefore, by theorem 4.3, we *can take the term-by-term derivatives*:

$$\frac{d}{dx}u(x,t) = \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n(t) \cos\left(n\pi \frac{x}{L}\right)$$

Now, I need to compute the u_{xx}, so consider:

1. u_{xx} is *assumed to be continuous*(hence piece-wise continuous)
2. u_x is an even function (cosine series), hence *no other requirements are needed*

Therefore, I am allowed to take term-by-term derivatives again:

$$u_{xx}(x,t) = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 B_n(t) \sin\left(n\pi \frac{x}{L}\right)$$

Lastly, I compute the derivative u_t, with the help of the thereom:

Theorem 4.4: Time Derivatives of Fourier Series

The Fourier series of a continous function $u(x,t)$:

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} \left[a_n(t) \cos\left(\frac{n\pi x}{L}\right) + b_n(t) \sin\left(n\pi \frac{x}{L}\right)\right]$$

can be **differentiated term by term with respective to t** if $\frac{\partial u}{\partial t}$ is **piece-wise smooth**

therefore, since I assumed u_t being continuous, I *take the term-by-term time derivatives again*:

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n'(t) \sin\left(n\pi \frac{x}{L}\right)$$

Step 2:

Now, I have computed all the derivatives, plugging back into the PDE:

$$B_n'(t) = -k\left(\frac{n\pi}{L}\right)^2 B_n(t)$$

this basically says:

$$B_n(t) = B_n(0)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

with $B(0) = B_n$ computed from the initial conditio:

$$f(x) = \sum_{n=1}^{\infty} B_n(0) \sin\left(n\pi \frac{x}{L}\right)$$

which is exactly the **coefficients of the sine series of $f(x)$**. Putting them back together:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi \frac{x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

4.4 Fourier Series with Inhomogeneous Heat Equation

Now, consider the case with the setup:

$$u_t = k u_{xx}$$

with B.C. and I.C.:

$$\begin{cases} u(0,t) = A(t) \\ u(L,t) = B(t) \\ u(x,0) = 0 \end{cases}$$

so that the **odd extension and the 2L extension will not be continuous**. Therefore, we cannot use term-by-term differentiation for $u(x,t)$.

In this case, the strategy is:

1. assume the *form of the solution being a series*
2. start from the u_{xx} and *compute backwards to $u(x,t)$*, so that no derivatives are involved

Ansatz: assume that the solution $u(x,t)$ takes the form of a *sine series*, then u_{xx} would also take the form of a sine series:

$$\begin{aligned} u_{xx} &= \sum_{n=1}^{\infty} \gamma_n(t) \sin\left(n\pi \frac{x}{L}\right) \\ \gamma_n(t) &= \frac{2}{L} \int_0^L u_{xx} \sin\left(n\pi \frac{x}{L}\right) dx. \end{aligned}$$

by the *orthogonality principle of* $\sin(...)$ *and* $\cos(...)$.

Step 1: Working backwards, I would like to **find γ_n as an expression of b_n**:

$$\begin{aligned} \gamma_n(t) &= \frac{2}{L} u_x \sin\left(n\pi \frac{x}{L}\right)\Big|_0^L - \frac{2}{L} \int_0^L u_x \frac{n\pi}{L} \cos\left(n\pi \frac{x}{L}\right) dx \\ &= -\frac{2}{L} \int_0^L u_x \frac{n\pi}{L} \cos\left(n\pi \frac{x}{L}\right) dx \\ &= -\frac{2}{L} \frac{n\pi}{L} u(x,t) \cos\left(n\frac{\pi x}{L}\right)\Big|_0^L - \frac{2}{L} \int_0^L \left(\frac{n\pi}{L}\right)^2 u(x,t) \sin\left(n\pi \frac{x}{L}\right) dx \\ &= \frac{2\pi n}{L^2} [A(t) - (-1)^n B(t)] - \left(\frac{n\pi}{L}\right)^2 b_n[u]. \end{aligned}$$

now, we have an expression of γ in terms of b_n.

Step 2: To figure out the expression of $u(x,t)$, assume that:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(n\pi \frac{x}{L}\right)$$

for some *unknown coefficients $b_n(t)$*. Now, taking its time derivative and plugging back in to the PDE with our **previous expression of $\gamma_n(t)$**:

$$
\begin{aligned}
b_n'(t) &= k\gamma_n(t) \\
b_n'(t) &= k\left(\frac{2\pi n}{L^2}[A(t) - (-1)^n B(t)] - \left(\frac{n\pi}{L}\right)^2 b_n[u]\right) \\
b_n'(t) + k\left(\frac{n\pi}{L}\right)^2 b_n(t) &= k\frac{2\pi n}{L^2}[A(t) - (-1)^n B(t)] \\
b_n'(t) + k\left(\frac{n\pi}{L}\right)^2 b_n(t) &= S_n(t).
\end{aligned}
$$

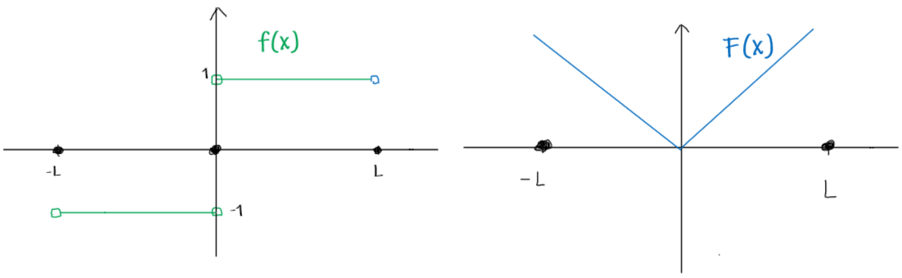
now, we get an linear ODE. Solving this basically means using an *integrating factor of $e^{-\int a dt}$*:

$$
\begin{aligned}
e^{-\left(\frac{n\pi}{L}\right)^2}\left(e^{k\left(\frac{n\pi}{L}\right)^2} b_n'(t)\right) &= S_n(t) \\
b_n(t) &= \int_0^L S_n(\tau) e^{k\left(\frac{n\pi}{L}\right)^2(\tau - t)}\, d\tau.
\end{aligned}
$$

which solves the PDE since now we know the $b_n(t)$.

4.5 Integrating Fourier Series

In short, since *integrating piece-wise smooth function $f(x)$ give you back a continuous function $F(x)$*



there is in general no **big problems on integrating term by term**.

Theorem 4.5: Integrating Fourier Series

Since Fourier Series are *piece-wise smooth*, the integration of it would be a smooth function. Therefore, the **term by term integration** will converge to the actual integration of the series.

- however, there is *no guarantee* that the integration of a Fourier Series is *still a Fourier Series*

***Example*: Term-By-Term Integration of Fourier Series**

Consider the series:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\pi \frac{x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\pi \frac{x}{L}\right)$$

since the integration will be *continuous*, the convergence will be **exact**:

$$\begin{aligned} F(x) &= a_0 x + \sum_{n=1}^{\infty} a_n \left(\frac{L}{n\pi}\right) \sin\left(n\pi \frac{x}{L}\right) + \sum_{n=1}^{\infty} b_n \left(\frac{L}{n\pi}\right) \left(1 - \cos\left(n\pi \frac{x}{L}\right)\right) \\ &= a_0 x + \sum_{n=1}^{\infty} b_n \frac{L}{n\pi} + \sum_{n=1}^{\infty} a_n \left(\frac{L}{n\pi}\right) \sin\left(n\pi \frac{x}{L}\right) \\ &\quad - \sum_{n=1}^{\infty} b_n \left(\frac{L}{n\pi}\right) \cos\left(n\pi \frac{x}{L}\right). \end{aligned}$$

which makes $F(x)$ **not a Fourier Series** due to the $a_0 x$ term, making it not periodic.

Therefore, **although the result still converges to the correct result**, if we need the Fourier Series property, we need to remove it:

$$G(x) = F(x) - a_0 x$$

4.6 Complex Notation of Fourier Series

Instead of sine and cosine, using the equivalence:

$$\begin{aligned} \cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned}$$

I can rewrite each *individual series as*:

$$a_n \cos\left(n\pi \frac{x}{L}\right) + b_n \cos\left(n\pi \frac{x}{L}\right) = \frac{e^{-in\pi \frac{x}{L}}}{2} (a_n + ib_n) + \frac{e^{in\pi \frac{x}{L}}}{2} (a_n - ib_n)$$

then, defining:

$$\begin{cases} c_n = a_n + ib_n, & n > 0 \\ c_n = a_n - ib_n, & n < 0 \\ c_0 = a_0, & n = 0 \end{cases}$$

we get **more compactly** (assuming we can directly take the derivative):

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi \frac{x}{L}}$$

and that

$$
\begin{aligned}
c_n[f] &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{i\frac{n\pi x}{L}}\,dx \\
c_n[f'] &= -i\frac{n\pi}{L}c_n[f] \\
& etc..
\end{aligned}
$$

where the orthogonality principle still holds, such that:

$$
\frac{1}{2L}\int_{-L}^{L} e^{i\frac{n\pi x}{L}}e^{-i\frac{m\pi x}{L}}\,dx = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}
$$

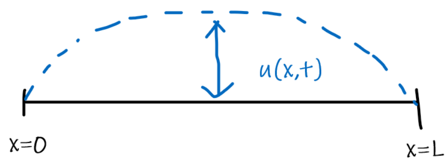
5 Wave Equation

This section discusses PDE related to waves such as:

- standing waves on vibrating string

5.1 One Dimensional Wave Equation

Consider a *string* of length L whose both ends are fixed:



then with $u(x, t)$ **indicating the displacement of string from equilibrium**, we have (omitting derivation):

$$u_{tt} = c^2 u_{xx}, \quad c^2 = \frac{T_0}{\rho_0}$$

where:

- T_0 means the *tension of the string* (assumed constant)
- ρ_0 means the *density* (also assumed constant)
- c mean the *speed of propagation of the opposite waves*

Theorem 5.1: Dirichlet's B.C. for 1-D Wave Equation

If we have the following setup:

$$u_{tt} = c^2 u_{xx}, \quad c^2 = \frac{T_0}{\rho_0} \tag{30}$$

where:

- T_0 means the *tension of the string* (assumed constant)
- ρ_0 means the *density* (also assumed constant)

and the **boundary conditions of**:

$$\begin{aligned} u(0, t) &= 0 \\ u(L, t) &= 0 \end{aligned}$$

.

since we have a **second order differential equation for t as well**, we need two initial conditions:

$$\begin{aligned} u(x,0) &= f(x), \quad \text{initial displacement} \\ u_t(x,0) &= g(x), \quad \text{initial velocity.} \end{aligned}$$

then the solution takes the form:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(n\pi\frac{x}{L}\right)\left(A_n \cos\left(n\pi\frac{ct}{L}\right) + B_n \sin\left(n\pi\frac{ct}{L}\right)\right)$$

and that using the *orthogonality*:

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\sin\left(n\pi\frac{x}{L}\right)dx \\ B_n &= \frac{2}{L}\left(\frac{L}{n\pi c}\right)\int_0^L g(x)\sin\left(n\pi\frac{x}{L}\right)dx. \end{aligned}$$

Reminder:

To solve PDEs we basically have introduced the single tool: separation of variables:

1. *separate $u(x,t)$ and substitute back to PDE to obtain two ODE*
2. *solve the ODE that works with boundary condition (usually homogeneous)*
3. *solve the form of the other ODE, and leaving the coefficients uncomputed*
4. *combine the solutions, and solve for the coefficients using the initial condition (usually inhomogeneous)*

Note:

Technically, there is the other technique of *starting from a Fourier Series and calculate the coefficients afterwards.* In the end, they will look the same. (e.g. see subsection 4.4)

Proof. With the setup mentioned in theorem 5.1, we first separate the variables:

Step 1: Let $u(x,t) = h(t)\varphi(x)$, substituting back to the PDE:

$$\frac{1}{c^2}\frac{h''}{h} = \frac{\varphi''}{\varphi} = -\lambda, \qquad \lambda > 0.$$

then we **obtained:**

$$\begin{aligned} \varphi'' + \lambda\varphi &= 0 \\ h'' + c^2\left(\frac{n\pi}{L}\right)^2 h &= 0. \end{aligned}$$

Step 2: Solve the ODE with boundary condition first. We have the boundary conditions of:

$$\varphi(0) = 0$$
$$\varphi(L) = 0$$
$$.$$

So, in this case, we obtain simply:

$$\varphi'' + \lambda\varphi = 0$$
$$\varphi_n x = \sin\left(n\pi\frac{x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

where the *coefficients are omitted here but added back later*.

Step 3: Solve the **form of the other ODE** with *inhomogenous conditions*:

$$h'' + c^2\left(\frac{n\pi}{L}\right)^2 h = 0$$

and we get:

$$h_n(t) = A_n \cos\left(n\pi\frac{ct}{L}\right) + B_n \sin\left(n\pi\frac{ct}{L}\right)$$

Step 4: combining back and **now solve for the coefficients**:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(n\pi\frac{x}{L}\right)\left(A_n \cos\left(n\pi\frac{ct}{L}\right) + B_n \sin\left(n\pi\frac{ct}{L}\right)\right)$$

then using the *inhomogenous conditions to compute the coefficients*

$$u(x,0) = f(x), \quad \text{initial displacement}$$
$$u_t(x,0) = g(x), \quad \text{initial velocity.}$$

we obtain:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\pi\frac{x}{L}\right)dx$$
$$B_n = \frac{2}{L}\left(\frac{L}{n\pi c}\right)\int_0^L g(x)\sin\left(n\pi\frac{x}{L}\right)dx.$$

□

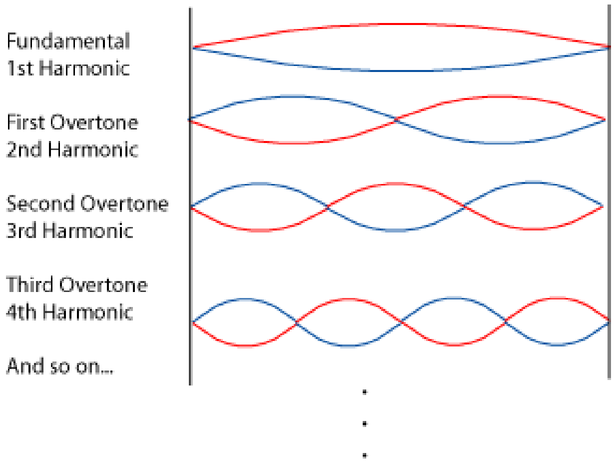
5.1.1 Interpretation of 1-D Wave Equation

In general, there are **two ways of interpretations** for the *standing wave solution of*

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(n\pi\frac{x}{L}\right)\left(A_n \cos\left(n\pi\frac{ct}{L}\right) + B_n \sin\left(n\pi\frac{ct}{L}\right)\right)$$

Interpretation 1: Standing wave

the term $\sin\left(n\pi\frac{x}{L}\right)$ represents the *shape of the wave*, and the rest is just an amplitude. Therefore, we basically have:



so we notice that:

- the number of *nodes* increases as n increases

Additionally, if we use the identity and rewrite:

$$A_n \cos\left(n\pi\frac{ct}{L}\right) + B_n \sin\left(n\pi\frac{ct}{L}\right) \to \sqrt{A_n^2 + B_n^2}\sin(\omega_n t + \theta_n)$$

where $\omega_n = \frac{n\pi c}{L}$ represents *angular frequency* and $\theta_n = \arctan\left(\frac{A_n}{B_n}\right)$ represents a phase (ignorable in this case), this means:

- the **frequency of oscillation is proportional to**:

$$\omega \propto \frac{c}{L} = \sqrt{\frac{T_0}{\rho_0}}\frac{1}{L}$$

Interpretation 2: Sum of two opposite waves

rewriting the equation to:

$$C_n \sin\left(n\pi\frac{x}{L}\right)\sin(\omega_n t + \theta_n) \to \frac{1}{2}C_n\left[\cos\left(\frac{n\pi}{L}(x-ct)\right) + \cos\left(\frac{n\pi}{L}(x+ct)\right)\right]$$

we see two *oppositely travelling waves*, each with **propagation speed** $c = \sqrt{\frac{T_0}{\rho_0}}$, and that:

- the right travelling wave:

$$\cos\left(\frac{n\pi}{L}(x-ct)\right)$$

solves the PDE $u_{tt} = c^2 u_{xx}$. And so does the left travelling wave.

- the left and right travelling wave independently **does not satisfy the initial condition**, but the **sum of them does**.

6 Sturm-Louisville Problems

This section deals with the problems in the form:

$$\frac{d}{dx}\left[p(x)\,\frac{d}{dx}u(x)\right] + (q(x) + \lambda\sigma(x))\;u(x) = 0.$$

and sometimes $\varphi(x)$ is used instead of $u(x)$, because we will have:

- $u(x,t) = \varphi(x)G(t)$
- hence the ODE will be on $\varphi(x)$

Definition 6.1: Regular Sturm-Louisville Equation

The equation

$$\frac{d}{dx}\left[p(x)\,\frac{d}{dx}\varphi(x)\right] + (q(x) + \lambda\sigma(x))\;\varphi(x) = 0.$$

is a **regular Sturm-Louisville Equation** if:

1. $p(x), q(x), \sigma(x)$ are real functions
2. $p(x), \sigma(x) > 0$
3. $a \leq x \leq b$ is the domain
4. the *boundary conditions are in the form*:

$$\begin{aligned} x = a, &\qquad \beta_1\varphi(a) + \beta_2\varphi'(a) = 0 \\ x = b, &\qquad \beta_3\varphi(b) + \beta_4\varphi'(b) = 0. \end{aligned}$$

   for $\beta_1, \beta_2, \beta_3, \beta_4$ being given

Note:

Note that if we take $p(x) = 1, q(x) = 0, \sigma(x) = 1$, we get back:

$$\varphi'' + \lambda\varphi = 0$$

which is the *simple ODE* that we have dealt with alot.

Reminder:

> *From the ODE lectures: Comparison between Linear Algebra and Sturm Liouville Theory*
>
> *(1) For a matrix A, we can consider the homogeneous system $Ax = 0$.*
> *In Sturm-Liouville theory, we consider the homogeneous problem $\frac{d}{dt}[p(x)\,\varphi'(x)] + q(x)\,\varphi(x) = 0$, $\varphi(a) = 0$, $\varphi(b) = 0$.*
> *(2) For a matrix A, we may consider the inhomogeneous system $Ax = w$, where w is given.*
> *In Sturm-Liouville theory, we consider the inhomogeneous problem $\frac{d}{dt}[p(x)\,\varphi'(x)] + q(x)\,\varphi(x) = f$, $\varphi(a) = A$, $\varphi(b) = B$, where f is a given function and A, B are given numbers.*
> *(3) A number λ is an eigenvalue of a matrix A if the system $Ax - \lambda x = 0$ has a non-trivial solution.*
> *A number λ is an eigenvalue of a Sturm-Liouville system if the problem $\frac{d}{dt}[p(x)\,\varphi'(x)] + (q(x) + \lambda\,\sigma(x))\,\varphi(x) = 0$, $\varphi(a) = 0$, $\varphi(b) = 0$ has a non-trivial solution.*
> *(4) The eigenvalues of a matrix A satisfy the algebraic equation $\det(A - \lambda I) = 0$.*
> *The eigenvalues of a Sturm-Liouville system satisfy the transcendental equation $u_\lambda(b) = 0$.*
> *(5) A matrix has finitely many eigenvalues.*
> *We will show that a Sturm-Liouville system has infinitely many eigenvalues.*
> *(6) For a symmetric matrix, eigenvectors corresponding to different eigenvalues are orthogonal.*
> *It turns out that, for a Sturm-Liouville system, eigenfunctions corresponding to different eigenvalues are orthogonal in an L^2-sense (to be made precise later).*
> *(7) Given a symmetric matrix, we can decompose every given vector into eigenvectors.*
> *Given a Sturm-Liouville system, it turns out that we can decompose every given function into eigenfunctions (generalization of Fourier series.)*

6.1 Regular Sturm-Louisville and Heat Equation

The Sturm-Louisville Equation can be used for solving non-uniform heat equation we have derived in 2.2:

$$c(x)\rho(x)u_t = (k(x)u_x)_x + Q$$

for a source term *proportional to $u(x,t)$* which represents the temperature here:

$$Q = \alpha(x)u(x,t)$$

and separating the variables $u(x,t) = \varphi(x)G(t)$ will give:

$$\frac{G'}{G} = \frac{1}{c\rho}\frac{(k\,\varphi')'}{\varphi} + \frac{\alpha}{c\rho} = -\lambda.$$

then we get **two ODEs**:

$$G' + \lambda G = 0$$
$$(k\,\varphi')' + \alpha\varphi + \lambda c\rho\varphi = 0.$$

and that the difficult to solve ODE is in the **Sturm-Louisville form**:

$$(k\,\varphi')' + (\alpha + \lambda c\rho)\varphi = 0$$

Theorem 6.1: Useful Properties of Regular Sturm-Liousville Problem

If we have a *Regular Sturm-Liousville Problem*, then we have the **following properties for the solution**:

1. λ will be real
2. infinite many eigenvalues, and they are monotonously increasing:

   $$\lambda_1 < \lambda_2 < \lambda_3 < ...$$

   such that there *exists a smallest eigenvalue but not a largest one*
3. Each eigenvalue λ_n has its correspoding eigenfunction $\varphi_n(x)$, and each eigenfunction has *exactly $n-1$ zeroes in the interval $a < x < b$* (comparison principle)
4. the *set of solutions $\{\varphi_n\}$* is *complete*:

   $$f(x) \approx \sum_{n=1}^{\infty} a_n \varphi_n(x)$$

   the sign becomes $=$ if $f(x)$ equals to the average of left and right limits at any jumps (recall Fourier Series)
5. the solutoin φ_n are *orthogonal to each other* **with respect to $\sigma(x)$**:

   $$\int_a^b \varphi_n(x)\varphi_m(x)\sigma(x)\,dx = 0, \quad \text{if } \lambda_n \neq \lambda_m$$

   therefore with initial condition $f(x)$:

   $$f(x) = \sum_{n=1}^{\infty} a_n \psi_n(x)$$
   $$a_n = \frac{\int_a^b f(x)\varphi_n(x)\sigma(x)\,dx}{\int_a^b \varphi_n^2(x)\sigma(x)\,dx}.$$

6. the **Rayleigh Quotient** *relates the eigenvalue and its eigenfunctions* by:

$$\lambda = \frac{-p\,\varphi\,\frac{d\varphi}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\varphi}{dx}\right)^2 - q\varphi^2\right]\,dx}{\int_a^b \varphi^2\sigma\,dx}$$

and the *linear algebra analogy* is, if a matrix A is symmetric:

$$RQ \equiv \frac{\vec{y}^T A\vec{y}}{\vec{y}^T\vec{y}}$$

and we see the connection with the above if $A\vec{y} = \lambda\vec{y}$:

$$RQ = \frac{\vec{y}^T\lambda\vec{y}}{\vec{y}^T\vec{y}} = \lambda$$

Note:

In the proof section, you will see that some properties, such as property 1 and property 5, falls out from the fact that **Sturm-Louisville Problem/Operator is Self-Adjoint/Hermitian**

Corollary 6.1: EigenValues for Heat Equations

Previously for Heat Equations, we had to solve the separated ODE:

$$\varphi'' + \lambda\varphi = 0$$

using **Rayleigh's Quotient**, we can show that the eigenvalues $\lambda \geq 0$. This is proven simply because this ODE is a *special case of Sturm-Liousville Equation*. Then, using Rayleigh Quotient:

$$\lambda = \frac{-\varphi\,\frac{d\varphi}{dx}\Big|_0^L + \int_0^L \left[\left(\frac{d\varphi}{dx}\right)^2\right]\,dx}{\int_0^L \varphi^2\,dx}$$

with either *Dirichlet or Neumann's B.C.*:

$$\lambda = \frac{\int_0^L \left[\left(\frac{d\varphi}{dx}\right)^2\right]\,dx}{\int_0^L \varphi^2\,dx} \geq 0$$

which completes the proof.

Additionally, this also shows there is a trivial solution for $\lambda = 0$ if we have a *Dirichlet's B.C.*:

$$\lambda = 0 = \int_0^L \varphi'^2 \, dx$$
$$\varphi = constant = 0.$$

6.2 Proofs For Regular Sturm-Louisville Properties

Before proving the 6 properties of theorem 6.1, first we need to know:

Definition 6.2: Linear Sturm-Liousville Operators

Let the Sturm-Louisville Equation has a **linear operator** $\mathcal{L}$:

$$\mathcal{L}[\varphi] = \frac{d}{dx}\left[p(x)\frac{d\varphi}{dx}\right] + q(x)\varphi \tag{31}$$

so that the **regular Sturm-Louisville Equation becomes**:

$$\mathcal{L}\left[\varphi\right] = -\lambda\sigma\varphi$$

Definition 6.3: Self-Adjoint/Hermitian Operators

An **operator** $\mathcal{L}$ is called **self-adjoint/hermitian** if it is *symmetric* in the sense that:

$$\int_a^b \left(u\,\mathcal{L}\left[v\right] - v\,\mathcal{L}\left[u\right]\right)\,dx = 0$$

for a **regular Sturm-Louisville Equation**, this is equivalent to saying:

$$\left.\left[u\frac{dv}{dx} - v\frac{du}{dx}\right]\right|_a^b = 0$$

which you will see it is **true automatically due to Sturm-Louisville B.C.**

In the end, the concept is an *analogy to linear algebra* of a **symmetric matrix A such that**:

$$A = A^t$$
$$x^T\left(A - A^T\right)y = 0$$
$$x^T Ay - y^T Ax = 0.$$

Note:

Hermitian/Self-Adjoint Operator is also mentioned in quantum mechanics:

If $\hat{O}$ is a hermitian, then

$$\int_{-\infty}^{\infty} f^*(x)\hat{O}\left[g(x)\right]\, dx = \int_{-\infty}^{\infty} \left(\hat{O}[f(x)]\right)^* g(x)\, dx$$

which is the same thing here because *everything is real*.

6.2.1 Proof for being Self-Adjoint/Hermitian

First, I prove that **the regular Sturm-Louisville Problem is self-adjoint**

Proof. Suppose u, v are solutions to the eqaution:

$$\begin{aligned} \mathcal{L}\left[u\right] &= -\lambda\sigma u \\ \mathcal{L}\left[v\right] &= -\lambda\sigma v \end{aligned}$$

.

and that they *satisfy the B.C. of the regular Sturm-Louisville Problem*.

Then, consider the quantity:

$$\begin{aligned} u\mathcal{L}\left[v\right] - v\mathcal{L}\left[u\right] &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p\frac{du}{dx}\right) \\ &= \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]. \end{aligned}$$

then, we need to take the intergal:

$$\int_a^b u\mathcal{L}[v] - v\mathcal{L}[u]\, dx = p\left[u\frac{dv}{dx} - v\frac{du}{dx}\right]\Bigg|_a^b$$

Now, consider the *Sturm-Louisville B.C.*: for any solution φ

$$\begin{aligned} x &= a, \quad \beta_1 u(a) + \beta_2 u'(a) = 0 \\ x &= b, \quad \beta_3 u(b) + \beta_4 u'(b) = 0. \end{aligned}$$

which says in this case, at $x = a$:

$$\begin{bmatrix}\beta_1 & \beta_2\end{bmatrix}\begin{bmatrix}u(a) \\ u'(a)\end{bmatrix} = 0 = \begin{bmatrix}\beta_1 & \beta_2\end{bmatrix}\begin{bmatrix}v(a) \\ v'(a)\end{bmatrix}$$

therefore, this means that the two vectors $\begin{bmatrix}u(a) \\ u'(a)\end{bmatrix}$ and $\begin{bmatrix}v(a) \\ v'(a)\end{bmatrix}$ is **parallel**. Hence:

$$det = \begin{vmatrix}u(a) & u'(a) \\ v(a) & v'(a)\end{vmatrix} = 0$$

Therefore:

$$u(a)v'(a) - v(a)u'(a) = 0$$

the same goes for $x = b$, and we just get:

$$\int_a^b u\mathcal{L}[v] - v\mathcal{L}[u]\, dx = p\left[u\frac{dv}{dx} - v\frac{du}{dx}\right]\Bigg|_a^b = 0$$

□

Note:

> It turns out that *Sturm-Louisville Equation* with **non-regular B.C.** such as the periodic condition:
>
> $$\begin{cases} \varphi(a) = \varphi(b) \\ p(a)\varphi'(a) = p(b)\varphi'(b) \end{cases}$$
>
> **also makes the Sturm-Louisville operator $\mathcal{L}$ self-adjoint**. Therefore, **self-adjoint/hermitian problems** are **wider** Sturm-Louisville.

6.2.2 Proof for Property 5

Now, I prove that the **eigenfunctions of a Self-Adjoint Problem (including Sturm-Louisville) are orthogonal** (*property 5 of theorem 6.1*) .

Proof. Consider two eigenfunctions of a Self-Adjoint (e.g. the Sturm-Louisville):

$$\mathcal{L}\left[\varphi_n\right] = -\lambda_n \sigma \varphi_n$$
$$\mathcal{L}[\varphi_m] = -\lambda_m \sigma \varphi_m.$$

since I know the operator is self-adjoint:

$$\int_a^b \varphi_n \mathcal{L}[\varphi_m] - \varphi_m \mathcal{L}[\varphi_n]\, dx = 0$$

therefore:

$$\int_a^b \varphi_n \mathcal{L}[\varphi_m] - \varphi_m \mathcal{L}[\varphi_n]\, dx = 0$$
$$\int_a^b \left(-\varphi_n \lambda_m \varphi_m \sigma + \varphi_m \lambda_n \sigma \lambda_n\right)\, dx = 0$$
$$(\lambda_n - \lambda_m) \int_a^b \sigma \varphi_n \varphi_m\, dx = 0.$$

so I get

$$\int_a^b \varphi_n(x)\varphi_m(x)\sigma(x)\, dx = 0, \quad \text{if } \lambda_n \neq \lambda_m$$

which only **depended on the self-adjoint operator and the form of an eigen "equation"** $\mathcal{L}[\varphi] = \lambda\sigma\varphi$ □

6.2.3 Proof for Property 1

Now, I prove *property 1 of theorem 6.1*: the eigenvalues of Self-Adjoint Problem/Sturm Louisville Problem is **real**:

Proof. Consider:

$$\mathcal{L}[\varphi] = -\lambda\sigma\varphi$$

and taking the conjugate:

$$\mathcal{L}[\varphi^*] = -\lambda^* \sigma \varphi^*$$

we *want to show that $\lambda = \lambda^*$*.

Consider the trick again using **self-adjoint** :

$$\int_a^b \varphi^* \mathcal{L}[\varphi] - \varphi L[\varphi^*] \, dx = 0$$
$$\int_a^b -\varphi^* \lambda \sigma \varphi + \varphi \lambda^* \sigma \varphi^* \, dx = 0$$
$$(\lambda^* - \lambda) \int_a^b \sigma \varphi^* \varphi \, dx = 0$$
$$(\lambda^* - \lambda) \int_a^b \sigma |\varphi|^2 \, dx = 0.$$

but since $|\varphi|^2 \geq 0$, and that $\sigma > 0$ for regular Sturm-Louisville Equation, this implies that **λ is real** ☐

Note:

> Again, the above two property depended mainly on:
>
> 1. an **eigen equation** with a **self-adjoint operator** $\mathcal{L}$. (essentially the property of a Hilbert Space)
>
> which is much more *general than Sturm-Louisville Problem.*

6.2.4 Proof for Property 6

Now I prove how the Rayleigh Quotient works in Sturm-Louisville Equation.

Proof. Begin with the equation itself:

$$\frac{d}{dx}\left(p\frac{d\varphi}{dx}\right) + q\varphi + \lambda\sigma\varphi = 0, \quad a < x < b$$
$$\mathcal{L}[\varphi] + \lambda\sigma\varphi = 0.$$

Now **multiplying eigenfunction φ and integrating** :

$$\int_a^b \varphi \mathcal{L}[\varphi] + \lambda\sigma\varphi^2 \, dx = 0$$

Note:

> The idea of multiplying by φ and taking integral comes from *linear algebra again*, where if we have the equation:
>
> $$Av = \lambda v$$
>
> then I *multiply the eigenvector v^T* on both side to get λ :
>
> $$v^T A v = \lambda v^T v$$
> $$\lambda = \frac{v^T A v}{v^T v}.$$

Then, continuing from above, I get:

$$\lambda = \frac{-\int_a^b \varphi \mathcal{L}[\varphi]\,dx}{\int_a^b \sigma\varphi^2\,dx}$$

Last but not least computing the quantity in the numerator using *integration by parts*:

$$\begin{aligned}
\int_a^b \varphi\mathcal{L}[\varphi]\,dx &= \int_a^b \varphi\left(\frac{d}{dx}\left[p\frac{d\varphi}{dx}\right]\right)\,dx + \int_a^b q\varphi^2\,dx \\
&= p\varphi\,\frac{d\varphi}{dx}\bigg|_a^b - \int_a^b p\left(\frac{d\varphi}{dx}\right)^2\,dx + \int_a^b q\varphi^2\,dx \\
&= p\varphi\,\frac{d\varphi}{dx}\bigg|_a^b - \int_a^b p\left(\frac{d\varphi}{dx}\right)^2 - q\varphi^2\,dx.
\end{aligned}$$

therefore, I obtain:

$$\lambda = \frac{-p\,\varphi\,\frac{d\varphi}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\varphi}{dx}\right)^2 - q\varphi^2\right]\,dx}{\int_a^b \varphi^2\sigma\,dx}$$

□

6.3 More on Rayleigh Quotient

Recall that

$$\lambda = \frac{-p\,\varphi\,\frac{d\varphi}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\varphi}{dx}\right)^2 - q\varphi^2\right]\,dx}{\int_a^b \varphi^2\sigma\,dx}$$

we see then $\lambda \geq 0$ if we know:

1. this is true: $-p\varphi\,\frac{d\varphi}{dx}\Big|_a^b \geq 0$
2. and this is true: $q(x) \leq 0$

Reminder:

> *The origin of discuss was the heat equation*
>
> $$(k\,\varphi')' + (\alpha + \lambda c\rho)\varphi = 0$$
>
> *so we see that $\alpha = q(x)$* in the regular S-L equation. *Therefore, this means that if we have:*
>
> $$Q = \alpha u \leq 0 = \textit{releasing heat}$$
>
> *then $\lambda \geq 0$ and at least we get a time dependence of $h(t) = e^{-\lambda t}$ to be* decaying

Theorem 6.2: Boundary Term for Rayleight Quotient

I refer the boundary term here as:

$$-p\varphi \left.\frac{d\varphi}{dx}\right|_a^b$$

and that **this is** 0 (easy to see) for B.C.:

$$\begin{cases} \varphi(a) = \varphi(b) = 0 & \text{Dirichlet} \\ \frac{d\varphi}{dx}|_a = \left.\frac{d\varphi}{dx}\right|_b = 0 & \text{Neumann} \\ \begin{cases} \varphi(a) = \varphi(b) \\ p(a)\varphi'(a) = p(b)\varphi'(b) \end{cases} & \text{Periodic} \end{cases}$$

and this **is** ≥ 0 **for the general regular S-L boundary condition**:

$$\beta_1\varphi(a) + \beta_2\varphi'(a) = 0$$
$$\beta_3\varphi(b) + \beta_4\varphi'(b) = 0.$$

if:

- β_1, β_2 **have different sign**
- β_3, β_4 **have the same sign**

Proof. This can be proved by basically computing explicitly:

$$\begin{aligned} -p\varphi \left.\frac{d\varphi}{dx}\right|_a^b &= p(a)\varphi(a)\frac{-\beta_1}{\beta_2}\varphi(a) - p(b)\varphi(b)\frac{-\beta_3}{\beta_4}\varphi(b) \\ &= -p(a)\frac{\beta_1}{\beta_2}\varphi^2(a) + p(b)\frac{\beta_3}{\beta_4}\varphi^2(b). \end{aligned}$$

therefore, if we need this to be ≥ 0, we need:

- β_1, β_2 have a *different sign*
- β_3, β_4 have *same sign*

□

6.3.1 Minimization Principle

In general, the equation:

$$\mathcal{L}[\varphi] = -\lambda\sigma\varphi$$

is difficult to solve. However, there are ways to *estimate the solution* (e.g. comparison principle from ODE).

Theorem 6.3: Minimization Principle

Consider the S-L problem with some boundary condition

$$\beta_1\varphi(a) + \beta_2\varphi'(a) = 0$$
$$\beta_3\varphi(b) + \beta_4\varphi'(b) = 0.$$

and that a *function u satisfies such boundary condition* (but may not need to satisfy the S-L equation). Then it must be **true for that u**:

$$\lambda_{\text{min of }\varphi} \leq \frac{-p\,u\,\frac{du}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{du}{dx}\right)^2 - qu^2\right]\,dx}{\int_a^b u^2\sigma\,dx}$$

more importantly: **minimum value of the Rayleigh quotient for all continuous functions** satisfying the boundary conditions (but not necessarily the differential equation) **is the lowest eigenvalue**

$$\lambda_1(\varphi) = \min RQ(u), \quad \forall u$$

This means that you can *estimate/approach* the minimum eigenvalue by using some easier functions u that *also satisfy the B.C.*

Proof. This proof uses the fact that the **eigenfunctions of a S-L equation forms a complete set**.

Consider the quantity $RQ(u)$ for some u:

$$RQ(u) = \frac{-\int_a^b u\mathcal{L}[u]\,dx}{\int_a^b u^2\sigma\,dx}$$

since the φ eigenfunctions would *form a complete set*:

$$u(x) = \sum_{n=1}^{\infty} a_n\varphi_n(x)$$

therefore, since the operator is linear:

$$\mathcal{L}[u] = \sum_{n=1}^{\infty} a_m\mathcal{L}[\varphi_m] = -\sum_{n=1}^{\infty} a_m\lambda_m\varphi_m$$

Then, computing the quantity:

$$
\begin{aligned}
-\int_a^b u\mathcal{L}[u]\,dx &= \int_a^b \sum_{n=1}^{\infty} a_n\varphi_n \sum_{m=1}^{\infty} a_m\lambda_m\varphi_m(x)\,dx \\
&= \int_a^b \sum_{n=1}^{\infty} a_n^2\lambda_n\sigma\varphi_n^2\,dx \\
&\geq \int_a^b \lambda_1 \sum_{n=1}^{\infty} a_n^2\sigma\varphi_n^2\,dx.
\end{aligned}
$$

due to *orthogonality of eigenfunctions*.

Last but not least, the denominator:

$$\int_a^b u^2\sigma\,dx = \int_a^b \sum_{n=1}^{\infty} a_n^2\varphi_n^2(x)\sigma\,dx$$

Combining, I get:

$$\frac{-\int_a^b u\mathcal{L}[u]\,dx}{\int_a^b u^2\sigma\,dx} \geq \lambda_1$$

□

Corollary 6.2: Parceval's Relation

This is also used a lot in QM. Consider the quantity for any u:

$$\int_a^b u^2\sigma\,dx = \int_a^b \sum_{n=1}^{\infty} a_n^2\varphi_n^2(x)\sigma\,dx$$

if we have a *normalized eigenfunction φ*, such that:

$$\int_a^b \varphi^2\sigma\,dx = 1$$

then we simplify to:

$$\int_a^b u^2\sigma\,dx = \sum_{n=1}^{\infty} a_n^2$$

Corollary 6.3: Bessel's Inequality

Again, for any u, if we **truncate the series** in corollary 6.2, then:

$$\int_a^b u^2\sigma\,dx \geq \sum_{n=1}^{M} a_n^2$$

for some $M < \infty$

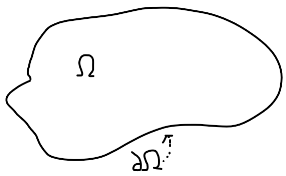
7 3D Heat Equation

Previously, we have been dealing with mostly 1D PDEs, for instance:

- *wave euation* $u_{tt} = c^2 u_{xx}$
- *heat equation* $u_t = k u_{xx}$

Now, we want to deal with *PDE in 3D*

For $\vec{x}$ inside the domain Ω:



Consider:

Theorem 7.1: 3D Heat Equation

The PDE for **3D Heat Equation** with $k = 1$:

$$u_t = \nabla^2 u$$

and we have the **boundary conditions of:**

$$a(\vec{x})u(\vec{x}, t) + b(\vec{x})\frac{\partial u}{\partial \vec{n}} = 0, \quad \vec{x} \in \partial\Omega.$$

where:

- $a(\vec{x}), b(\vec{x}) \geq 0$ but *cannot both be zero*
- the term $\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \hat{n}$ is the *normal derivative*

and the **initial condition of**:

$$u(\vec{x}, 0) = f(\vec{x})$$

then, **separating the variable** $u = G(t)\varphi(\vec{x})$. we get:

$$\frac{G'}{G} = \frac{\nabla^2 \varphi}{\varphi} \equiv -\lambda$$

therefore, we obtain $G = e^{-\lambda t}$ and the **time independent equation**:

$$\begin{cases} \nabla^2 \varphi + \lambda \varphi = 0, & \text{the PDE} \\ a\varphi + b\frac{\partial \varphi}{\partial n} = 0, & \text{the B.C.} \end{cases}$$

this equation:

$$\nabla^2\varphi + \lambda\varphi = 0$$

is called the **Helmholtz Equation**

7.1 Properties of Helmholtz Equation

Essentially, solving the 3D problem becomes solving for the **Helmholtz Equation**. In fact, you will see that it shares *similar attributes with the Sturm-Liousville Problem*.

Theorem 7.2: Properties of Helmholtz Equation

Properties of the below equation include:

$$\nabla^2\varphi + \lambda\varphi = 0$$

1. λ is **real**
2. there exists a **smallest eigenvalue** but *no largest eigenvalue* such that:
$$\lambda_1 < \lambda_2 \le \lambda_3 \le ... \le \lambda_n$$
notice that we are having $\le$ because there could be *repeated eigenvalues*
3. Corresponding to an eigenvalue, there may be **many eigenfunctions** since the eigenvalues can be repeated (but from there, you can construct linearly independent ones)

   Additionally, those eigenvalues can only be **repeated for finite number of times**
4. the eigenfunctions **form a complete set**, such that any piecewise smooth function can be represented as:
$$f(\vec{x}) = \sum_{n=1}^{\infty} a_n[f]\varphi_n(\vec{x})$$
5. Eigenfunctions of **different eigenvalues** are **orthogonal** over the region Ω, so that:
$$\int_\Omega \varphi_n(\vec{x})\varphi_m(\vec{x})d\vec{x} = 0, \quad \text{if } \lambda_n \neq \lambda_m$$
notice that this is the same as S-L theorem, but with $\sigma = 1$
6. The eigenvalue is related to the eigenfunction by **Rayleigh Quotient**:
$$\lambda = \frac{-\oint_{\partial\Omega} \varphi\frac{\partial\varphi}{\partial n}dS + \Omega\left|\nabla\varphi\right|^2 d\vec{x}}{\int_\Omega \varphi(\vec{x})^2 d\vec{x}}$$

Proof for Property 6. The proof of **property 6** is the most straight-forward (*same idea for S-L problem and the linear algebra*). Consider:

$$\vec{\nabla}\cdot\left(\varphi\vec{\nabla}\varphi\right) = \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + \varphi\nabla^2\varphi$$

then, consider the *original PDE*:

$$\begin{aligned}
\nabla^2\varphi + \lambda\varphi &= 0 \\
\varphi\left(\nabla^2\varphi + \lambda\varphi\right) &= 0 \\
\vec{\nabla}\cdot\left(\varphi\vec{\nabla}\varphi\right) - \left|\vec{\nabla}\varphi\right|^2 + \lambda\varphi^2 &= 0 \\
-\lambda\int_\Omega \varphi^2 d\vec{x} &= \oint_{\partial\Omega}\varphi\vec{\nabla}\varphi\cdot\hat{n}dS - \int_\Omega \left|\vec{\nabla}\varphi\right|^2 d\vec{x} \\
\lambda &= \frac{-\oint_{\partial\Omega}\varphi\frac{\partial\varphi}{\partial n}dS + \int_\Omega \left|\nabla\varphi\right|^2 d\vec{x}}{\int_\Omega \varphi^2 d\vec{x}}.
\end{aligned}$$

□

Interestingly, we will see that $\lambda \geq 0$ **for most of the cases**:

1. for the **general Helmholtz B.C.**:

$$a\varphi + b\frac{\partial\varphi}{\partial n} = 0,\ \vec{x}\in\partial\Omega$$

and that $a, b > 0$, then I have:

$$\frac{a}{b}\varphi = -\frac{\partial\varphi}{\partial n},\quad \vec{x}\in\partial\Omega$$

Since I know:

$$\lambda = \frac{-\oint_{\partial\Omega}\varphi\frac{\partial\varphi}{\partial n}dS + \int_\Omega \left|\nabla\varphi\right|^2 d\vec{x}}{\int_\Omega \varphi^2 d\vec{x}}$$

then I just need to consider the term:

$$-\oint_{\partial\Omega}\varphi\frac{\partial\varphi}{\partial n}dS = \oint_{\partial\Omega}\frac{a}{b}\varphi^2 dS \geq 0$$

hence already:

$$\lambda \geq 0$$

if $\lambda = 0$ (we want to know if $\lambda > 0$ would be true), then the term above also needs to be zero:

$$\varphi = 0,\quad \vec{x}\in\partial\Omega$$

but this means that the *Helmholtz equation becomes*

$$\nabla^2\varphi + 0 = 0$$

which becomes the **Laplace's Equation**, and you can recall the **Min and Max principle** from theorem 3.5, that the *minimum and maximum of φ inside a region can only exists on the boundary*.

Therefore, sine $\varphi = 0,\ x\in\partial\Omega$, this means that *inside the region*:

$$\varphi = 0,\ \vec{x}\in\Omega$$

Therefore, it means that **strictly** $\lambda > 0$ if $a, b > 0$.

2. if we have **Dirichlet B.C.**, we can use the previous part:

$$a\varphi + b\frac{\partial \varphi}{\partial n} = 0,\ \vec{x} \in \partial\Omega$$

but with $b = 0$. This means that:

$$\varphi = 0, \quad \vec{x} \in \partial\Omega$$

so that:

$$\lambda = \frac{-\oint_{\partial\Omega} \varphi \frac{\partial \varphi}{\partial n} dS + \int_{\Omega} |\nabla\varphi|^2 \, d\vec{x}}{\int_{\Omega} \varphi^2 d\vec{x}} = \frac{\int_{\Omega} |\nabla\varphi|^2 \, d\vec{x}}{\int_{\Omega} \varphi^2 d\vec{x}} \geq 0$$

then consider *if* $\lambda = 0$, this means:

$$|\nabla\varphi|^2 = 0$$

this means that $\varphi = \vec{c}$, a constant vector. However, since Dirichlet B.C. tells me:

$$\varphi = 0, \quad \vec{x} \in \partial\Omega$$

therefore I obtain the trivial solution for $\lambda = 0$. Hence:

$$\lambda > 0$$

3. if we have **Neumann B.C.**, I just have:

$$\frac{\partial \varphi}{\partial n} = 0, \quad \vec{x} \in \partial\Omega$$

this tells me that:

$$\lambda = \frac{-\oint_{\partial\Omega} \varphi \frac{\partial \varphi}{\partial n} dS + \int_{\Omega} |\nabla\varphi|^2 \, d\vec{x}}{\int_{\Omega} \varphi^2 d\vec{x}} = \frac{\int_{\Omega} |\nabla\varphi|^2 \, d\vec{x}}{\int_{\Omega} \varphi^2 d\vec{x}} \geq 0$$

and if $\lambda = 0$, I have the same scenario:

$$|\nabla\varphi|^2 = 0$$

but since $\varphi = \vec{c}$ is *allowed for any arbitrary constant*. Hence, it is possible for $\lambda = 0$ to have a non-trivial solution.

Theorem 7.3: Eigenvalue for Helmhotlz Equation

From the above, we can see that with **Helmholtz Equation**:

$$\nabla^2\varphi + \lambda\varphi = 0$$

1. $\lambda > 0$ if we have **Dirichlet B.C.**
2. $\lambda \geq 0$ if we have **Neumann B.C.**
3. $\lambda > 0$ if $a, b > 0$ for the **mixed condition**:

$$a\varphi + b\frac{\partial \varphi}{\partial n} = 0,\ \vec{x} \in \partial\Omega$$

Theorem 7.4: Self-Adjoint/Hermitian Operator

The Helmholtz equation can be written as:

$$\hat{H}[\varphi] = -\lambda\varphi, \quad \hat{H} = \nabla^2$$

and this operator ∇^2 is **Hermitian/Self-Adjoint**

Note:

Being a Hermitian leads to many other useful properties, which you shall see soon (*recall Hilbert Space in QM*):

1. *orthogonality of eigenfunctions*
2. etc.

Proof for Theorem 7.4. For being Hermitian Operator, we need:

$$\int_\Omega \left(u\nabla^2 v - v\nabla^2 u\right) d\vec{x} = 0$$

for u, v being solutions to the *Helmholtz Equation*.

To show this, first consider:

$$\begin{aligned}\nabla \cdot (u\nabla v) &= \vec{\nabla} u \cdot \vec{\nabla} v + u\nabla^2 v \\ \nabla \cdot (v\nabla u) &= \vec{\nabla} v \cdot \vec{\nabla} u + v\nabla^2 u.\end{aligned}$$

Therefore, I get:

$$\vec{\nabla} \cdot (u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u$$

this hints at *using the Divergence Theorem*:

$$\begin{aligned}\int_\Omega \left(u\nabla^2 v - v\nabla^2 u\right) d\vec{x} &= \oint_{\partial\Omega} (u\nabla v - v\nabla u) \cdot \hat{n} dS \\ &= \oint_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS.\end{aligned}$$

since both u, v satisfy the Helmholtz B.C.:

$$\begin{bmatrix} u & \frac{\partial u}{\partial n} \\ v & \frac{\partial v}{\partial n} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because I know that a, b are not both zero:

$$\begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This means that the *column vectors are linearly dependent*:

$$\det \begin{bmatrix} u & \frac{\partial u}{\partial n} \\ v & \frac{\partial v}{\partial n} \end{bmatrix} = 0$$

Therefore:

$$u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} = 0$$

so that I obtain:

$$\int_\Omega \left(u\nabla^2 v - v\nabla^2 u\right) d\vec{x} = \oint_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS = 0$$

□

7.1.1 Proof for Property 5

In summary, this proof evinces itself because the ∇^2 operator for the *eigen-equation* is **Hermitian**:

$$\nabla^2\varphi = -\lambda\varphi$$

Proof for Property 5. Consider the two solutions φ_n, φ_m. Since ∇^2 is *Hermitian* :

$$\int_\Omega \left(\varphi_m\nabla^2\varphi_n - \varphi_n\nabla^2\varphi_m\right) = 0$$

then using the **eigen-equation**:

$$\int_\Omega \left(\varphi_n(-\lambda_n\varphi_n) - \varphi_m(-\lambda_m\varphi_m)\right) d\vec{x} = 0$$

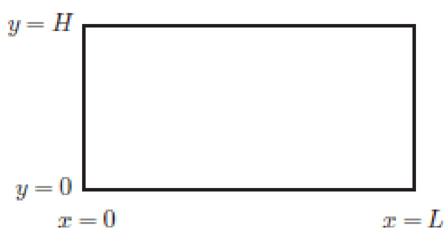
$$(\lambda_m - \lambda_n)\int_\Omega \varphi_m\varphi_n \, d\vec{x} = 0.$$

Therefore, **if $\lambda_m \neq \lambda_n$, φ_n, φ_m are orthogonal** □

7.2 2D Rectangular Drum

Eventually, you will see that this problem *ended up solving the Helmholtz Equation.*

Consider a drum with:



and we want to measure *displacement u* with the *2D Wave Equation*:

$$u_{tt} = c^2\left(u_{xx} - u_{yy}\right)$$

with Boundary Condition:

$$\begin{cases} u(x,0,t) = 0 \\ u(x,H,t) = 0 \\ u(0,y,t) = 0 \\ u(L,y,t) = 0 \end{cases}$$

the *initial condition is:*

$$\begin{cases} u(x,y,0) = 0 \\ u_t(x,y,0) = 0 \end{cases}$$

Theorem 7.5: Solution of 2D Rectanguar Drum

The solution to the **above setup** is:

$$\varphi_{n,m} = \sin\left(n\pi\frac{x}{L}\right)\sin\left(m\pi\frac{y}{H}\right)$$

as the **eigenfunctions**, and

$$\lambda_{n,m} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

The **overall solutions** looks like:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\varphi_{mn}\left[A_{mn}\cos\left(c\sqrt{\lambda}t\right) + B_{mn}\sin\left(c\sqrt{\lambda}t\right)\right]$$

since we know the **orthogonality condition**:

$$\int_0^L\int_0^H \varphi_{mn}\varphi_{m'n'}\,dx\,dy = 0, \quad \text{if } m \neq m' \text{ or } n \neq n'.$$

Therefore, I can solve for **both coefficients using inhomogenous initial condition**:

$$A_{mn} = \frac{\int_0^L\int_0^H \varphi_{mn}\alpha(x,y)\,dx\,dy}{\int_0^L\int_0^H \varphi_{mn}^2\,dx\,dy}.$$

similarly, for the other coefficient:

$$c\sqrt{\lambda_{mn}}B_{mn} = \frac{\int_0^L\int_0^H \varphi_{mn}\beta(x,y)\,dx\,dy}{\int_0^L\int_0^H \varphi_{mn}^2\,dx\,dy}$$

Reminder:

Recall that the above solution ***conforms to the multi-dimension solution in Quantum Mechanics***

Proof. The procedure is the *same as solving previous PDE*:

1. **separating the variable and obtain ODEs:**

$$u = \varphi(x, y)h(t)$$

then the PDE becomes:

$$\begin{cases} h'' + \lambda c^2 h = 0 \\ \nabla^2 \varphi + \lambda \varphi = 0 \end{cases}$$

hence I got:

$$h(t) = A \sin\left(c\sqrt{\lambda} t\right) + B \cos\left(c\sqrt{\lambda} t\right)$$

and the **Helmholtz Equation**:

$$\nabla^2 \varphi + \lambda \varphi = 0$$

To solve the **Helmholtz Equation**, apply *another separation of variables*:

$$\varphi(x, y) = f(x)g(y)$$

hence:

$$\frac{f''}{f} = -\frac{g''}{g} - \lambda = -\mu$$

and I obtain the ODEs:

$$\begin{cases} f'' + \mu f = 0 \\ g'' + (\lambda - \mu)g = 0 \end{cases}$$

2. **using the homogeneous condition to simplify solution/find eigenfunctions**:

$$\begin{cases} f(0) = 0 \\ f(L) = 0 \end{cases}$$

hence I get:

$$f_n = \sin\left(n\pi \frac{x}{L}\right), \quad \mu_n = \left(\frac{n\pi}{L}\right)^2$$

for $n = 1, 2, \ldots$.

Similarly:

$$\begin{cases} g(0) = 0 \\ g(H) = 0 \end{cases}$$

therefore, by *viewing $\lambda - \mu$* as a whole:

$$g_m = \sin\left(m\pi \frac{y}{H}\right), \quad \lambda - \mu_n = \left(\frac{m\pi}{H}\right)^2$$

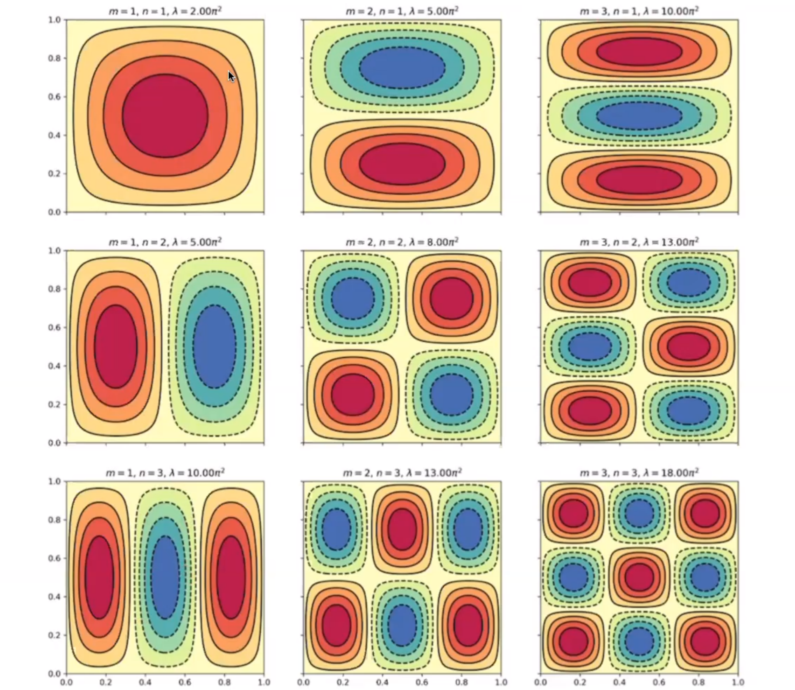
3. **putting the solutions together and use the inhomogeneous condition**:

$$\varphi = f(x)g(y) = \sin\left(n\pi \frac{x}{L}\right) \sin\left(m\pi \frac{y}{H}\right) = \varphi_{n,m}$$

and

$$\lambda = \mu_n + \left(\frac{m\pi}{H}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 = \lambda_{n,m}$$

at this point, we can already see **degeneracies** (e.g. same λ but different combination of eigenfunctions). This will be more evident if $H = L$:



where *blue indicates negative u and red indicates positive u*

Now, the **overall solutions** looks like:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\left(m\pi\frac{x}{L}\right)\sin\left(n\pi\frac{x}{L}\right)\left[A_{mn}\cos\left(c\sqrt{\lambda}t\right) + B_{mn}\sin\left(c\sqrt{\lambda}t\right)\right]$$

since we know the **orthogonality condition**:

$$\varphi_{mn} = \sin\left(n\pi\frac{x}{L}\right)\sin\left(m\pi\frac{x}{L}\right)$$

$$\int_0^L\int_0^H \varphi_{mn}\varphi_{m'n'}\,dx\,dy = 0, \quad \text{if } m \neq m' \text{ or } n \neq n'.$$

Therefore, I can solve for **both coefficients using inhomogenous initial condition**:

$$u(x,y,0) = \alpha(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\varphi_{mn}$$

$$A_{mn} = \frac{\int_0^L\int_0^H \varphi_{mn}\alpha(x,y)\,dx\,dy}{\int_0^L\int_0^H \varphi_{mn}^2\,dx\,dy}.$$

similarly, for the other coefficient:

$$c\sqrt{\lambda_{mn}}B_{mn} = \frac{\int_0^L\int_0^H \varphi_{mn}\beta(x,y)\,dx\,dy}{\int_0^L\int_0^H \varphi_{mn}^2\,dx\,dy}$$

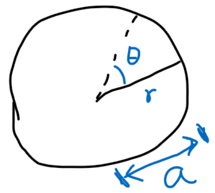
Note:

> Notice that **two families of coefficients** A_{mn}, B_{mn} can be determined from **one initial condition** since we see there are *two eigenfunctions each each eigenvalue* λ_{mn}

□

7.3 Circular Drum

Now, consider the setup:



where we have a circular drum of **radius** a, and the following equations:

PDE:

$$u_{tt} = c^2 \nabla^2 u$$

in polar coordinate $u(r, \theta, t)$, and

B.C.:

$$u(a, \theta, t) = 0$$

I.C.:

$$u(r, \theta, 0) = \alpha(r, \theta)$$
$$u_t(r, \theta, 0) = \beta(r, \theta).$$

Theorem 7.6: Solution of Circular Drum

The hidden *continuity constraints are*:

$$u(r, \pi, t) = u(r, -\pi, t)$$
$$u_t(r, \pi, t) = u_t(r, -\pi, t).$$

and that:

$$|u(0, \theta, t)| < \infty$$

The overall **solution** of the above setup looks like:

$$u(r, \theta, t) = J_m(\sqrt{\lambda_{mn}}\, r) \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} \begin{Bmatrix} \cos\left(c\sqrt{\lambda_{mn}}\, t\right) \\ \sin\left(c\sqrt{\lambda_{mn}}\, t\right) \end{Bmatrix}.$$

so that we have basically the double sum of the following:

$$\begin{cases} A_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \cos(m\theta) \cos\left(c\sqrt{\lambda_{mn}} t\right) \\ B_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \sin(m\theta) \cos\left(c\sqrt{\lambda_{mn}} t\right) \\ C_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \cos(m\theta) \sin\left(c\sqrt{\lambda_{mn}} t\right) \\ D_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \sin(m\theta) \sin\left(c\sqrt{\lambda_{mn}} t\right) \end{cases}.$$

where:

- there are in total **four product terms, with four coefficients**
- the $J_m(z)$, $z \equiv \sqrt{\lambda}\, r$ is the **Bessel Function of the 1st kind of Order** m, $m = 0, 1, 2...$
- the function $J_m(\sqrt{\lambda_{mn}}\, r)$ is **orthogonal to each other for same** m**, different** n

and the eigenvalues are:

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2, \quad m = 0, 1, 2...$$

where:

- z_{mn} denotes the **n-th zero root of the Bessel Function** $J_m(z)$

Lastly, the coefficients are then solved by using *orthogonalities*:

$$u(r, \theta, 0) = \alpha(r, \theta) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \left[A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta)\right]$$
$$A_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^a \alpha(r, \theta)\, J_m\left(\sqrt{\lambda_{mn}}\, r\right) \cos(m\theta) r\, dr\, d\theta}{\int_{-\pi}^{\pi} \int_0^a J_m^2\left(\sqrt{\lambda_{mn}}\, r\right) \cos^2(m\theta) r\, dr\, d\theta}.$$

the same idea goes for the other three coefficients.

Before proving/solving this problem, several properties you will need to know:

Theorem 7.7: Property of a Singular S-L Equation

The Sturm-Louisville equation:

$$(rf')' - \frac{m^2}{r} f + r\lambda f = 0$$

for $0 \le r \le a$, $m = 0, 1, 2, \ldots$ does **not satisfy the Regular S-L condition**, but is a **Singular S-L Equation**.

Then Singular S-L, as it turns out, has the **same property as Regular S-L Equation**:

1. λ is real
2. infinite number of eigenvalues
3. orthogonality of eigenfunctions with respect to $\sigma = r$:

   $$\int_0^a f_{mn} f_{mn'} r \, dr = 0, \quad \text{if } n \ne n'$$

   and notice that m **is just a fixed parameter at a time**, n is the varying thing.
4. etc.

Theorem 7.8: Bessel Equation and Bessel Functions

The **Bessel Equation** looks like:

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + \left(z^2 - m^2\right) f = 0$$

and there are **two linearly independent solutions**:

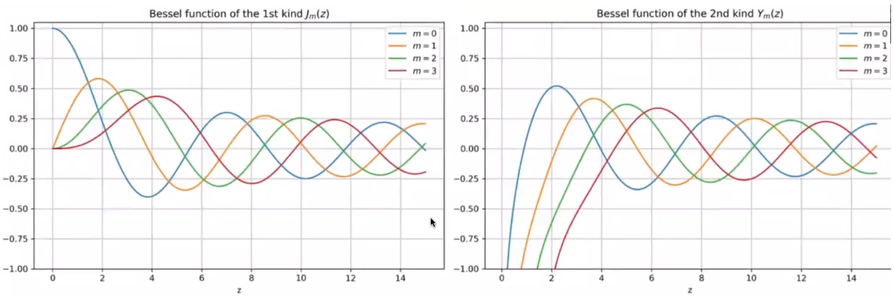
- $J_m(z)$ is the **Bessel Function of the 1st Kind of Order** m
- $Y_m(z)$ is the **Bessel Function of the 2nd Kind of Order** m

for $m = 0, 1, 2, \ldots$, and both of which *are not expressible in terms of elementary functions*. However, there is some *relatively accurate approximations*:

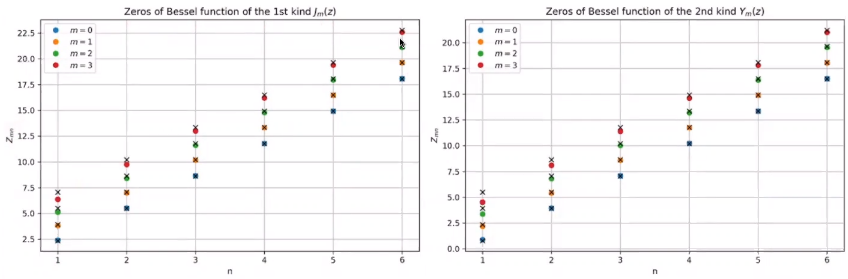
$$J_m(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right), \quad \text{as } z \to \infty$$
$$Y_m(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right), \quad \text{as } z \to \infty$$

.

Graphically, the two Bessel functions look like:



where **four orders** are plotted for each Bessel Function.
Additionally, the *zero-root of the approximation function v.s. Bessel Function look like*



where **4 orders of both Bessel Functions** are plotted :

- horizontal axis plots the *n-th root*
- vertical axis plots the value of the root
- $\times$ means the root of the *approximation*

Note:

Notice that the **root for the same order m looks linear**, as it is of the form:

$$n \times (\text{some constant})$$

which resembles the zeroes of the sinusoidal $(\sin(z))$:

$$n \times \pi$$

Now, we can go to the proof and solve the PDE:

Proof. Again, the same idea:

1. start with separating the variable to attempt to reduce the PDE to ODE:

$$\begin{cases} h'' + \lambda c^2 h = 0, \quad h = \begin{Bmatrix} \sin\left(c\sqrt{\lambda}t\right) \\ \cos\left(c\sqrt{\lambda}t\right) \end{Bmatrix} \\ \nabla^2 \varphi + \lambda\varphi = 0 \end{cases}$$

Now, arriving at the *Helmholtz Equation*, we first rewrite it in *polar coordinates*:

$$\frac{1}{r}(r\varphi_r)_r + \frac{1}{r^2}\varphi_{\theta\theta} + \lambda\varphi = 0$$

separating the variables again into ODE, by letting $\varphi = f(r)g(\theta)$:

$$\frac{1}{r}(rf')'g + \frac{1}{r^2}fg'' + \lambda fg = 0$$
$$-\frac{g''}{g} = r\frac{(rf')'}{f} + \lambda r^2 \equiv \mu.$$

where this μ would be non-negative because I want the *obvious solution g to be oscillatory*. Then the ODEs I obtained are:

$$g'' + \mu g = 0$$
$$r(rf')' + (\lambda r^2 - \mu)f = 0.$$

2. now, solving the ODEs using *homogeneous conditions first*

$$g'' + \mu g = 0$$
$$g(\theta) = c_1 \sin(\sqrt{\mu}\theta) + c_2 \cos(\sqrt{\mu}\theta).$$

using the *continuity constraint* :

$$\begin{cases} g(-\pi) = g(\pi) \\ g'(-\pi) = g'(\pi) \end{cases}$$

I got:

$$\begin{cases} -c_1 \sin\left(\sqrt{\mu}\pi\right) = c_1 \sin\left(\sqrt{\mu}\pi\right) \\ -c_2 \sin\left(\sqrt{\mu}\pi\right) = c_2 \sin\left(\sqrt{\mu}\pi\right) \end{cases}$$

since I know that *it cannot be the trivial solution of $c_1 = c_2 = 0$*, this means that:

$$\mu = m^2, \quad m = 1, 2, 3...$$

Now, consider if $\mu = 0$. This means the ODE becomes:

$$g'' = 0$$
$$g(\theta) = c_1\theta + c_2.$$

from the same *continuity constraint*, I obtain:

$$g(\theta) = c_2$$

Then I get in total:

$$g(\theta) = c_1 \sin(\sqrt{\mu}\theta) + c_2 \cos(\sqrt{\mu}\theta), \quad \mu = 0, 1, 2...$$

Now, the hard part is solving for **f(r)**, first, notice that we can convert the ODE to a **Sturm-Louisville Equation**:

$$(rf')' - \frac{m^2}{r}f + r\lambda f = 0$$

where I have:

- $p = r, q = \frac{m}{r^2}, \sigma = r$ as the S-L Equation
- since $0 \leq r \leq a$, it is **not a regular Sturm-Louisville Equation** (e.g. regular S-L needs $p, \sigma > 0$)
- this is known as a **Singular Sturm-Louisville Equation** (which turns out to have the same property as Regular S-L)

Then, rearranging the equation:

$$r^2 f'' + r f' + \left(\lambda r^2 - m^2\right) f = 0$$

and the B.C.:

$$\begin{cases} f(a) = 0 \\ |f(0)| < \infty \end{cases}$$

letting $z = r\sqrt{\lambda}$ (*recall that $\lambda > 0$ as a property of Regular S-L with periodic B.C.*), I have:

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + \left(z^2 - m^2\right) f = 0$$

which is a **Bessel Equation**, and that the B.C. becomes:

$$\begin{cases} f(\sqrt{\lambda}a) = 0 \\ |f(0)| < \infty \end{cases}$$

to make sure the condition $|f(0)| < \infty$ is satisfied, **consider $z \to 0$** for both Bessel Functions:

$$J_m(z) \sim \begin{cases} 1 & m = 0 \\ \frac{1}{2^m m!} z^m \to 0 & m > 0 \end{cases}$$

$$Y_m(z) \sim \begin{cases} \frac{2}{\pi} \ln(z) \to -\infty & m = 0 \\ -\frac{2^m (m-1)!}{\pi} z^{-m} \to -\infty & m > 0 \end{cases}.$$

Therefore, only $J_m(z)$ works. This means that:

$$f(z) = c_1 J_m(z) = c_1 J_m(\sqrt{\lambda}\, r)$$

now, *applying the other boundary condition and solving for λ*:

$$J_m(\sqrt{\lambda}\, a) = 0$$

There is an infinite number of zeros of each Bessel function $J_m(z)$. Let **z_{mn} designate the nth zero of $J_m(z)$**:

$$\sqrt{\lambda} a = z_{mn}$$
$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2.$$

now, again we see λ_{mn} **depending on two independent parameters:**

- m signifying the **m-th order of Bessel Function**, coming from the solution of $\mu = m^2$

- n signifying the **n-th zero root** of the Bessel Function $J_m(z)$

Finally, because **eigenfunctions of Singular S-L are orthogonal wrt σ**, and that $\sigma = r$ here:

$$\int_0^a J_m(\sqrt{\lambda_{mn}}\, r) \cdot J_m(\sqrt{\lambda_{mn'}}\, r) r\, dr = 0$$

for a given m, but different $n \neq n'$ (since the eigenfunction itself is $J_m(z)$).

3. Lastly, **combining all the eigenfunctions**:

$$\begin{aligned} u(r,\theta,t) &= f(r)g(\theta)h(t) \\ &= J_m(\sqrt{\lambda_{mn}}\, r) \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} \begin{Bmatrix} \cos\left(c\sqrt{\lambda_{mn}}\, t\right) \\ \sin\left(c\sqrt{\lambda_{mn}}\, t\right) \end{Bmatrix}. \end{aligned}$$

where there will be **4 products => 4 coefficients**.

$$\begin{cases} A_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \cos(m\theta) \cos\left(c\sqrt{\lambda_{mn}} t\right) \\ B_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \sin(m\theta) \cos\left(c\sqrt{\lambda_{mn}} t\right) \\ C_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \cos(m\theta) \sin\left(c\sqrt{\lambda_{mn}} t\right) \\ D_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \sin(m\theta) \sin\left(c\sqrt{\lambda_{mn}} t\right) \end{cases}.$$

Now, I just need to **determine the coefficients** using the *inhomogeneous initial conditions*:

$$\begin{aligned} u(r,\theta,0) = \alpha(r,\theta) &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \left[A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta)\right] \\ A_{mn} &= \frac{\int_{-\pi}^{\pi} \int_0^a \alpha(r,\theta) J_m\left(\sqrt{\lambda_{mn}}\, r\right) \cos(m\theta) r\, dr\, d\theta}{\int_{-\pi}^{\pi} \int_0^a J_m^2\left(\sqrt{\lambda_{mn}}\, r\right) \cos^2(m\theta) r\, dr\, d\theta}. \end{aligned}$$

and the same for B_{mn} with $\sin(m\theta)$ instead. Notice that this works because:

(a) $\cos(m\theta)$ is *orthogonal* to $\cos(n\theta)$ for $m \neq n$

(b) $\cos(...)$ is *orthogonal* to $\sin(...)$

(c) J_m is *orthogonal* to J_n

Then, to get C_{mn} and D_{mn}, just use the condition:

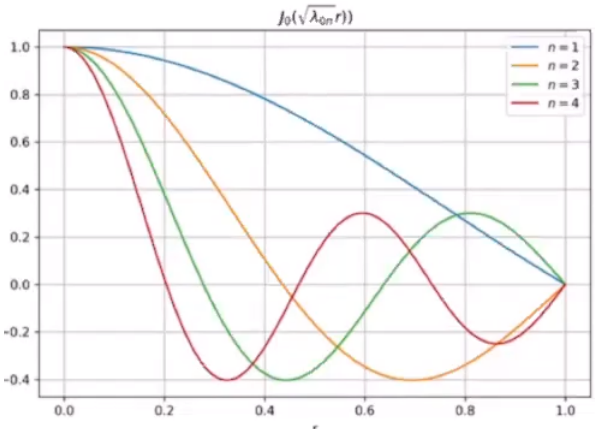
$$u_t(r,\theta,0) = \beta(r,\theta)$$

so that only C_{mn}, D_{mn} terms are left. Then just use the orthogonalaties.

□

Last but not least, some **related plots**:

Figure 1: $J_0\left(\sqrt{\lambda_{0n}}\, r\right)$, for $n = 1, 2, 3, 4$

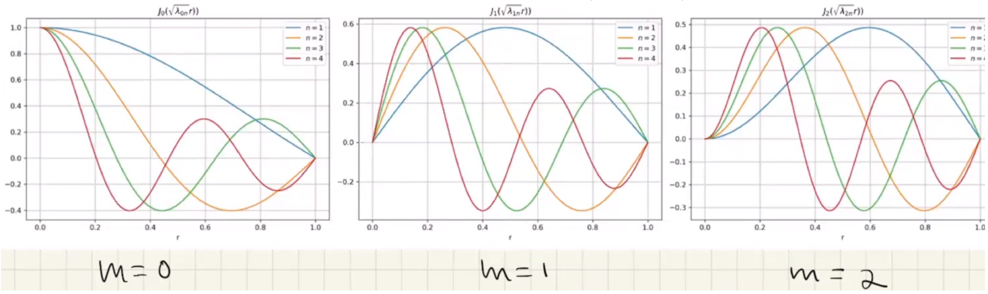


where we see:

- for each n, there are $n - 1$ zeroes in the range $[0, a)$.

Similarly, if we are increasing m:

Figure 2: Increasing m for $J_m\left(\sqrt{\lambda_{mn}}\, r\right)$

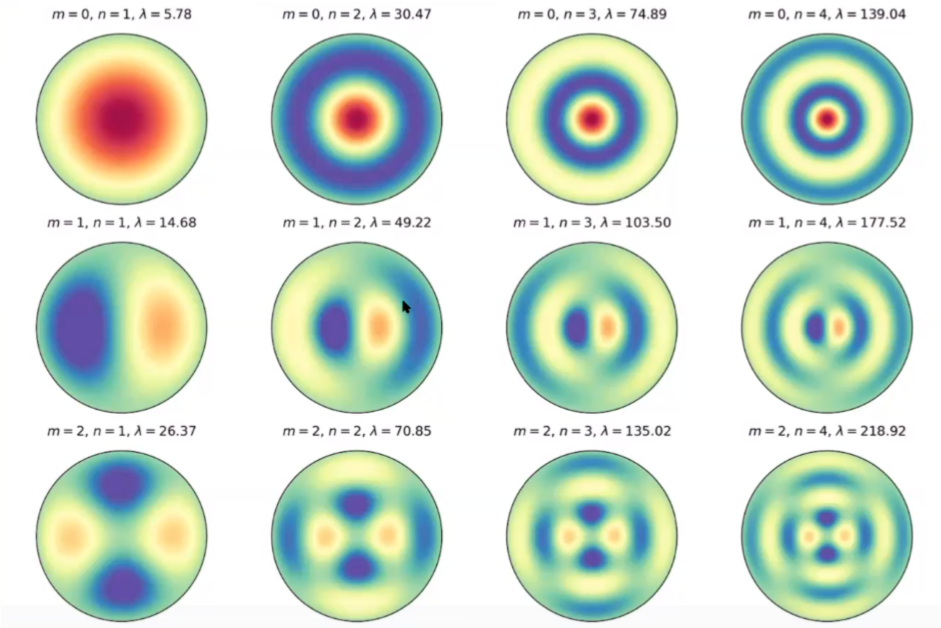


where:

- we see that it looks more and more like a *cosine function* as m increases, but number of zeroes is still controlled by n

Some plots of the solution looks like:

Figure 3: u with only the $\cos(m\theta)$ terms

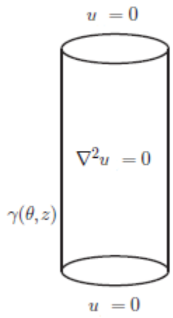


where we see:

- for $m = 0$ (first row in graph), there is no θ dependence since $\cos(0 \cdot \theta) = 1$
- for $m \neq 0$, we have the θ variations coming
- as n increases (per row), we see *more zeroes/wiggles* due to $J_m\left(\sqrt{\lambda_{mn}}\, r\right)$
- as m increases (per column), we see increase in *symmetries* due to $\cos(m\theta)$ term

7.4 Cylindrical Laplace's Equation

In the end, it still becomes solving some *sort of a Bessel Equation*



with a **radius of** R, and a **height of** H, and we have the following setup:

- the PDE is the Laplace Equation (*no time dependence here*):

$$\nabla^2 u = 0$$

  and we are going to use *cylindrical coordinates*

- the B.C. is the following:

$$\begin{cases} u(r, \theta, 0) = 0 \\ u(r, \theta, H) = 0 \\ u(R, \theta, z) = \gamma(\theta, z) \end{cases}$$

- no I.C. since it is steady state

Before going to solve the problem, something you need to know:

Theorem 7.9: Modified Bessel Equation and Bessel Functions

The **Modified Bessel Equation** looks like:

$$\omega^2 \frac{d^2 f}{d\omega^2} + \omega \frac{df}{d\omega} + -\left(\omega^2 + m^2\right) f = 0$$

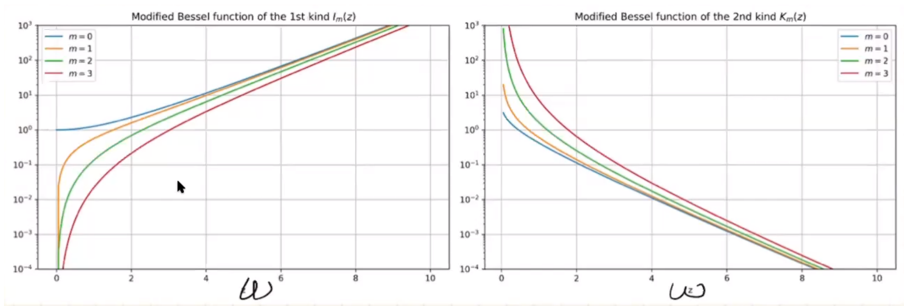
notice that we know have $-\left(\omega^2 + m^2\right)$ instead of the normal version $+\left(z^2 - m^2\right)$ and there are **two linearly independent solutions**:

- $I_m(\omega) = I_m\left(n\pi \frac{r}{H}\right)$ is the **Bessel Function of the 1st Kind of Order** m

- $K_m(\omega) = K_m\left(n\pi \frac{r}{H}\right)$ is the **Bessel Function of the 2nd Kind of Order** m

for $m = 0, 1, 2, ...$, and both of which *are not expressible in terms of elementary functions*.

Graphically, we have:



where we see that:

- $I_m(\omega)$ is *bounded at $\omega = 0$*, and goes to ∞ as $\omega \to \infty$
- $K_m(\omega)$ is bounded at ∞, but *unbounded at $\omega = 0$*
- notice that the above plot uses **log scale**. So we are having exponential growth for I_m

Theorem 7.10: Solution to Cylindrical Laplace's Equation

The solution to the above setup is:

$$u(r, \theta, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} I_m \left(n\pi \frac{r}{H}\right) \sin \left(n\pi \frac{z}{H}\right) [A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta)]$$

where notice that we have:

- $I_m\left(n\pi \frac{r}{H}\right)$ having a **known eigenvalue $\lambda = \frac{n\pi}{H}$**. This is because we have Laplace Equation and the B.C. helped us solved:

$$h'' + \lambda h = 0$$
$$h_n(z) = \sin\left(n\pi \frac{z}{H}\right), \quad \lambda_n = \frac{n\pi}{H}.$$

  for $u = \varphi(r, \theta) h(z)$

then, solving for the coefficients using orthogonality:

$$A_{mn} = \frac{\int_0^H \int_{-\pi}^{\pi} \gamma(\theta, z) \sin\left(n\pi \frac{z}{H}\right) \cos(m\theta)\, d\theta\, dz}{I_m(n\pi \frac{R}{H}) \int_0^H \int_{-\pi}^{\pi} \sin^2\left(n\pi \frac{z}{H}\right) \cos^2(m\theta)\, d\theta\, dz}$$

where notice that $I_m\left(n\pi \frac{R}{H}\right)$ will just be a number

and the same procedure can be applied to B_{mn}

Proof. Same procedure in general:

1. **separating the variables and obtain ODEs:**

   Let $u = \varphi(r,\theta), h(z)$, and the Laplacian in Cylindrical Coordinate becomes:

   $$\nabla^2 u = u_{zz} + \frac{1}{r}\left(ru_r\right)_r + \frac{1}{r^2}u_{\theta\theta}$$

   then *inserting our separated variables*:

   $$\frac{h''}{h} = -\frac{\nabla^2_{r,\theta}\varphi}{\varphi} \equiv -\lambda$$

   we picked **$-\lambda$ because then we have a solution for $h(z)$:**

   $$h'' + \lambda h = 0$$

   with B.C.:

   $$h(0) = h(H) = 0$$

   hence we obtain the solution:

   $$h_n(z) = \sin\left(n\pi\frac{z}{H}\right), \quad \lambda_n = \left(\frac{n\pi}{H}\right)^2$$

   and **notice that λ has already been solved here!**. Then similarly:

   $$\nabla^2_{r,\theta}\varphi = \lambda\varphi$$

   and notice that this **is not Helmholz Equatino**, which would be $\nabla^2\varphi + \lambda\varphi = 0$ for $\lambda > 0$. Hence, we need to *separate variables again:*
   let $\varphi(r,\theta) = f(r)g(\theta)$:

   $$\frac{1}{r}\frac{\left(rf'\right)'}{f} + \frac{1}{r^2}\frac{g''}{g} - \lambda = 0$$
   $$r\frac{\left(rf'\right)'}{f} - \lambda r^2 = -\frac{g''}{g} \equiv \mu.$$

   now, I picked $\mu \geq 0$ because then we can solve g to be periodic.

2. **solving the ODEs with homogeneous B.C.**, first solving $g(\theta)$:

   $$g'' + \mu g = 0$$

   then, we have the *periodic B.C. which we have done before*. The solution is:

   $$g(\theta) = c_1 \sin(m\theta) + c_2\cos(m\theta), \quad \mu = m^2$$

   for $m = 0, 1, 2, ...$
   More complicated is the $f(r)$ part:

   $$\left(rf'\right)' - \lambda rf - \frac{\mu}{r}f = 0$$
   $$\left(rf'\right)' - \lambda rf - \frac{m^2}{r}f = 0.$$

for $\lambda = \left(\frac{n\pi}{H}\right)^2$, but notice that this is **not the Bessel Equation** due to a sign difference. By substituting:

$$\omega = \sqrt{\lambda} r = n\pi \frac{r}{H}$$

we get:

$$\omega^2 \frac{d^2 f}{d\omega^2} + \omega \frac{df}{d\omega} + -\left(\omega^2 + m^2\right) f = 0$$

which is the **Modified Bessel Equation** (theorem 7.9), and there are two solutions $I_m(\omega), K_m(\omega)$. However, for **boundedness at origin**, we would only have $I_m(\omega)$ left:

$$f(r) = I_m\left(n\pi \frac{r}{H}\right)$$

for $m = 0, 1, 2...$ being the *order of Bessel Function*, coming from $\mu = m^2$.

3. **Lastly, assembling the solutions and solve for coefficients using inhomogeneous boundary condition**:

$$\begin{aligned} u(r, \theta, z) &= f(r) g(\theta) h(z) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} I_m\left(n\pi \frac{r}{H}\right) \sin\left(n\pi \frac{z}{H}\right) \left[A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta)\right]. \end{aligned}$$

then using the *inhomgenous condition*:

$$u(R, \theta, z) = \gamma(\theta, z)$$

we get:

$$A_{mn} = \frac{\int_0^H \int_{-\pi}^{\pi} \gamma(\theta, z) \sin\left(n\pi \frac{z}{H}\right) \cos(m\theta)\, d\theta\, dz}{I_m(n\pi \frac{R}{H}) \int_0^H \int_{-\pi}^{\pi} \sin^2\left(n\pi \frac{z}{H}\right) \cos^2(m\theta)\, d\theta\, dz}$$

and the same idea goes for B_{mn}

□

Lastly, consider some related plots:

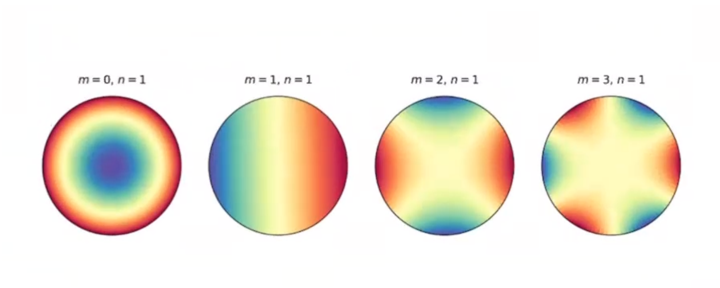
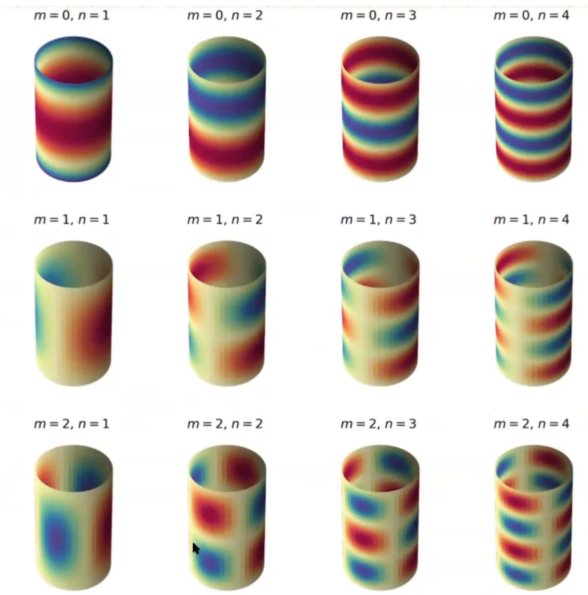


Figure 4: $I_m \sin\left(n\pi \frac{z}{H}\right) \cos\left(m\theta\right)$ for a fixed z, and for $n = 1$

where we see that:

- when $m = 0, \cos\left(m\theta\right) = 1$, hence there is *no θ dependence*
- as m increases, the *symmetries increases* due to the $\cos\left(m\theta\right)$ term

additionally:



where we see:

- in a row, as n increases, the number of *nodes/wiggles increases*, due to the $\sin\left(n\pi \frac{z}{H}\right)$ term
- in a column, as m increases, the number of *symmetries increases*. Same effect in previous plot.

7.5 Spherical Laplace's Equation

Note:

> Recall the property of **eigenfunctions in a Hilbert Space**. If they are forming a **complete set**, then that means **eigenfunctions of one PDE** could be used to **solve another PDE** if *they have the same geometry and B.C.*

Now, consider a *sphere* (e.g. approximating Earth)

with a radius of R, where we have the setup of:

- PDE:
$$\nabla^2 w + \lambda w = 0$$
for $w(\varphi, \theta)$ in *spherical coordinate* . Then the Laplacian becomes:
$$\nabla^2 w = \frac{1}{R^2}\frac{1}{\sin(\varphi)}(\sin(\varphi) w_\varphi)_+ \frac{1}{R^2}\frac{1}{\sin^2(\varphi)} w_{\theta\theta}$$
note that this is only considering ω on the surface.
- B.C. is then only *continuity equations*.

Theorem 7.11: Solution to Spherical Laplacian On Surface

If we only consider the PDE on the surface, i.e. $\omega(\theta, \varphi)$, then we have the solution:
$$\omega = \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} P_n^m(\cos(\varphi))$$
where we have:

- $P_n^m(x), x = \cos(\varphi)$ being the *Legendre Polynomial of First Kind*, for $n \geq m$

Theorem 7.12: Solution to Spherical Laplacian Inside

If we want to consider the situation, we have the setup:

- PDE:
$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$
- B.C. is:
$$u(R, \theta, \varphi, t) = 0$$

- I.C. is:

$$u(r,\theta,\varphi,0) = F(r,\theta,\varphi)$$
$$\frac{\partial u}{\partial t}(r,\theta,\varphi,0) = G(r,\theta,\varphi).$$

then the solution looks like:

$$u(e,\theta,\varphi,t) = \begin{Bmatrix} \cos\left(c\sqrt{\lambda}t\right) \\ \sin\left(c\sqrt{\lambda}t\right) \end{Bmatrix} \frac{1}{\sqrt{r}} J_{n+\frac{1}{2}}\left(\sqrt{\lambda}r\right) \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} P_n^m(\cos(\varphi))$$

where:

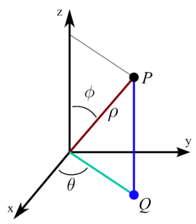
- $f(r) = 1/\sqrt{r}J_{n+\frac{1}{2}}\left(\sqrt{\lambda}r\right)$ is called the **Spherical Bessel Function**, which is *bounded at origin*
- for λ being the solution for the equation:

$$J_{n+\frac{1}{2}}\left(\sqrt{\lambda}R\right) = 0$$

- and for $n \geq m$, so that $\lambda R^2 = n(n+1)$

Lastly, if you want to use **orthgonality to solve for coefficients**, note that the **weights are** $d\theta, \sin(\varphi)d\varphi, r^2dr$ respectively

Just for reference, the following coordinate system is used:



proof for spherical laplacian on the surface. This solution *will be used anyway for the full solution*, so let us start with this.

1. **Separating the variables and obtaining ODEs**: let $w = q(\theta)g(\varphi)$, then we have:

$$\sin(\varphi)\frac{\sin(\varphi)g'}{g} + R^2\sin^2(\varphi)\lambda + \frac{q''}{q} = 0$$

the ODEs are:

$$\begin{cases} \sin(\varphi)\frac{\sin(\varphi)g'}{g} + R^2\sin^2(\varphi)\lambda = \mu \\ -\frac{q''}{q} = \mu \end{cases}$$

then for the q solution, we obviously have:

$$q = \begin{Bmatrix} \sin(m\theta) \\ \cos(m\theta) \end{Bmatrix}$$

for $m = 0, 1, 2...$

2. **Solving the ODEs using homogeneous conditions** , in this case, we just have the φ part left:

$$(\sin(\varphi)g')' + \left(R^2\lambda^2 \sin(\varphi) - \frac{m^2}{\sin(\varphi)}\right) g = 0$$

which is *not a regular S-L*, because:

- B.C. is only $|g(0)| < \infty, |g(\pi)| < \infty$
- and that $p = \sin(\varphi)$ is *not always* ≥ 0, and etc.

now, it turns out we can make it into a Spherical Bessel's Equation, by having $x = \cos(\varphi)$, and using $\sin^2(\varphi) = 1 - \cos^2(\varphi) = 1 - x^2$, we get:

$$\frac{d}{dx}\left[\left(1 - x^2\right)\frac{dg}{dx}\right] + \left(\lambda R - \frac{m}{1 - x^2}\right) g = 0$$

then this **only has a bounded solution when** :

- $\lambda R^2 = n(n+1)$
- when $n \geq m$
- the solution is $g_{mn} = P_n^m(x)$ being the *associate Legendre Polynomial of the First Kind*

there is no boundary condition, so we proceed.

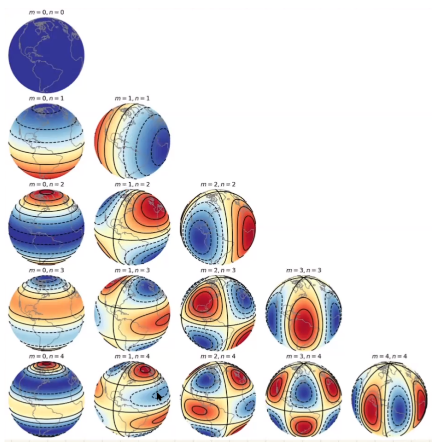
3. **lastly, combine**, we have:

$$w_{mn}(\varphi, \theta) = \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} P_n^m(\cos(\varphi))$$

for $m = 0, 1, 2, 3...$, and $n \geq m$

□

therefore, graphically, we have the **triangular shape due to $n >= m$**:



proof for the entire solution. If we want to solve for also the part *inside the entire sphere*, we would have the following separation:

$$u(r,\theta,\varphi,t) = f(r)\omega(\theta,\varphi)h(t)$$

which ends up having ODEs:

$$\frac{d^2h}{dt^2} = -\lambda c^2 h$$
$$\nabla^2 (f\omega) + \lambda (f\omega) = 0.$$

the solution for h is the ordinary ones:

$$h = \begin{Bmatrix} \sin\left(c\sqrt{\lambda}t\right) \\ \cos\left(c\sqrt{\lambda}t\right) \end{Bmatrix}$$

and splitting the other equation into parts: $\omega(\theta,\varphi) = q(\theta)g(\varphi)$ I have:

$$\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \left(\lambda r^2 - \mu\right) f = 0$$
$$\frac{d}{d\varphi}\left(\sin(\varphi)\frac{dg}{d\varphi}\right) + \left(\mu\sin(\varphi) - \frac{m^2}{\sin(\varphi)}g\right) = 0.$$

with the ODEs for q being skipped because it is simply:

$$q = \begin{Bmatrix} \sin(m\theta) \\ \cos(m\theta) \end{Bmatrix}$$

now, since we *don't know μ for the ODE with $f(r)$ yet*, start with the second one to solve, using the above theorem 7.12, I get:

$$g(x) = g(\cos(\varphi)) = P_n^m(x)$$

with the eigenvalues $\mu = n(n+1)$ for $n \geq m$

Lastly, solving for $f(r)$, we end up with:

$$\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \left(\lambda r^2 - n(n+1)\right) f = 0$$

which is a **Spherical Bessel Equation** , and the solution is:

$$f(r) = \frac{1}{\sqrt{r}} J_{n+\frac{1}{2}}\left(\sqrt{\lambda}r\right)$$

for boundedness at origin. □

8 Nonhomogeneous Problems

Now, consider the cases when we have some *sources or bath*, such that we have basically *no homogenous conditions left for us to simplify using separation of variables*. In this case, there are **some tricks that can convert non-homogenous parts into homogeneous parts**

8.1 Heat Flow with Sources and Nonhomogeneous Boundary Conditions

Theorem 8.1: No Source but Nonhomogenous BC

If we have the setup of:

- PDE:
$$u_t = ku_{xx}$$
- BC:
$$\begin{cases} u(0,t) = A \\ u(L,t) = B \\ u(x,0) = f(x) \end{cases}$$
where A, B are constants, *signifying a bath/reservoir term*.

then the trick is by letting:

$$v(x,t) = u(x,t) - u_{eq}$$

where u_{eq} is the **equilibrium solution**:

$$u_{eq} = A + \frac{B-A}{L}x$$

and then we will *easily solve v* having now **homogeneous boundary conditions**, ending up with:

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi \frac{x}{L}\right)$$

and the coefficients are:

$$B_n = \frac{1}{L}\int_0^L [f(x) - u_{eq}(x)] \sin\left(n\pi \frac{x}{L}\right) dx$$

Note:

Essentially, the idea is the same in physics: we want to **transform to the "center of mass frame"** which is the u_{eq}, and then solve for **displacement away from center of mass** which is $v(x,t) = u(x,t) - u_{eq}$

Proof. If we consider an equilibrium solution, then it **must satisfy**:

$$ku_{xx} = 0$$

and the **same B.C.** of:

$$\begin{cases} u_{eq}(0) = A \\ u_{eq}(L) = B \end{cases}$$

so obviously the solution is:

$$u_{eq} = A + \frac{B - A}{L} x$$

now, consider $v = u - u_{eq}$, we want to **reform the PDE and B.C. with v only**:

$$\begin{aligned} v_t = u_t &= ku_{xx} \\ &= ku_{xx} + 0 \\ &= k\left(u - u_{eq}\right)_{xx} \\ &= kv_{xx}. \end{aligned}$$

and the B.C. **becomes homogeneous**:

$$\begin{cases} v(0,t) = u(0,t) - u_{eq}(0) = 0 \\ v(L,t) = u(L,t) - u_{eq}(L) = 0 \end{cases}$$

then the solution for v is *obvious*:

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi \frac{x}{L}\right)$$

□

Theorem 8.2: Steady Source with Homogenous BC

If we have the setup of:

- PDE:

$$u_t = ku_{xx} + Q(x)$$

  which is *steady source as Q is not dependent on time*

- BC:

$$\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

then the trick is by letting (again):

$$v(x,t) = u(x,t) - u_{eq}$$

where u_{eq} is the **equilibrium solution** (if $Q(x) = 1$):

$$u_{eq} = \frac{1}{2k}x^2 + \frac{L}{2k}x$$

then we would end up with **homogeneous PDE** with the Q term cancelled, ending up with the same solution for v:

$$v(x,t) = \sum_{n=1}^{\infty} \sin\left(n\pi\frac{x}{L}\right)$$

and the same idea for coefficients.

Proof. Now, consider the same approach of u_{eq}. The idea is that *the equilibrium state should still exist*:

$$k u_{eqxx} = -Q(x)$$

integrating twice would solve the equation. To simplify this, let $Q(x) = 1$, then we have:

$$u_{eq}(x) = -\frac{1}{2k}x^2 + c_1 x + c_2$$

since this must **also solve the B.C.**, I hence have:

$$\begin{cases} c_2 = 0 \\ c_1 = \frac{L}{2k} \end{cases}$$

hence I get:

$$u_{eq} = -\frac{1}{2k}x^2 + \frac{L}{2k}x$$

now, we want to let $v = u - u_{eq}$ and **reform the PDEs**:

$$\begin{aligned} v_t = u_t &= k u_{xx} + Q \\ &= k u_{xx} - k u_{eq_{xx}} \\ &= k v_{xx}. \end{aligned}$$

which becomes *homogeneous*. The rest is trivial. □

Note:

If we have a **steady source but a non-homogenous BC**, we can just **combine the above two approaches**.

Theorem 8.3: Non-steady Sources and BC

If we have the setup of:

- PDE:

$$u_t = k u_{xx} + Q(x,t)$$

- BC:

$$\begin{cases} u(0,t) = A(t) \\ u(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$$

if we consider the same trick by letting **considering a reference solution** $r(x,t)$ that:

- just **satisfies the B.C.**
- but **does not satisfy** the PDE

with:

$$v(x,t) = u(x,t) - r(x,t)$$

note that we **do not have equilibrium solution anymore since conditions are time dependent**

Then we would end up with:

- **homogeneous B.C.**
- still **inhomogeneous PDE** of:

$$\begin{aligned} v_t = u_t - r_t &= ku_{xx} + Q(x,t) - r_t(x,t) \\ &= kv_{xx} + kr_{xx} + Q(x,t) - r_t(x,t) \\ &= kv_{xx} + \bar{Q}(x,t). \end{aligned}$$

to deal with this, we need **eigenfunction expansion**

Proof. Since we just need to consider $r(x,t)$ that solves B.C., we can consider the linear solution (it could be anything):

$$r(x,t) = A(t) + \frac{B(t) - A(t)}{L}x$$

which *solves the B.C. only*. Then, by letting:

$$v(x,t) = u(x,t) - r(x,t)$$

I have the *boundary conditions simplified*:

$$\begin{cases} v(0,t) = u(0,t) - r(0,t) = 0 \\ v(L,t) = 0 \end{cases}$$

but the PDE cannot be simplified since $r(x,t)$ only dealt with the B.C.:

$$\begin{aligned} v_t = u_t - r_t &= ku_{xx} + Q(x,t) - r_t(x,t) \\ &= kv_{xx} + kr_{xx} + Q(x,t) - r_t(x,t) \\ &= kv_{xx} + \bar{Q}(x,t). \end{aligned}$$

which is still inhomogenous. $\square$

8.2 Eigenfunction Expansion

In general, we see that we can still solve the problem relatively easily if we have:

- **steady** source Q and/or boundary conditions

If we have source and boundary being non-steady, we can still find a way to recede the B.C. to homogeneous case, but **not the source term**.

In this section, I will first discuss how to solve the unsteady source problem with:

- *homogeneous* B.C. using **Eigenfunction Expansion**
- *unsteady* B.C. using **Eigenfunction Expansion AND Green's Formula**

8.2.1 Eigenfunction Expansion

This section deals with the case when you have a **homogeneous B.C.**, but an **unsteady source** $Q(x, t)$.

Consider the problem we had before:

$$u_t = ku_{xx} + \bar{Q}(x, t)$$

with now the boundary condition:

$$\begin{cases} u(0, t) = 0 \\ u(L, t) = 0 \end{cases}$$

and the initial condition:

$$u(x, 0) = f(x)$$

Theorem 8.4: Solution to Homogenous B.C.

For the **setup mentioned above**, the solution is:

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x)$$

for $a_n(t)$ *represents the variable coefficient*:

$$a_n(t) = a_n(0)e^{-\lambda_n kt} + e^{-\lambda_n kt} \int_0^t \bar{q}_n(t)e^{\lambda_n kt}\, dr$$

where $\bar{q}_n(t)$ is defined to be:

$$\bar{Q}(x, t) = \sum_{n=1}^{\infty} \bar{q}_n(t)\varphi_n(x)$$

for $\varphi_n(x)$ being the **eigenfunction to he related/homogeneous**

version of the PDE, which is the solution to the below:

$$\omega_t = k\omega_{xx} + (0)$$
$$\varphi'' + \lambda\varphi = 0.$$

with **homogeneous B.C.** gives:

$$\varphi_n(x) = \sin\left(n\pi\frac{x}{L}\right), \; \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Proof. 1. First, find the **eigenfunction of the related homogeneous problem**. This would be

$$\omega_t = k\omega_{xx}.$$

Solving for ω we have the eigenfunction from the eigen-equation:

$$\varphi'' + \lambda\varphi = 0$$

with **homogeneous B.C.** gives:

$$\varphi_n(x) = \sin\left(n\pi\frac{x}{L}\right), \; \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

2. **Use eigenfunction expansion to write down the solution** in the form of:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(x)$$

which we can immediately solve $b_n(0)$ *for later use*:

$$b_n(0) = \frac{\int_0^L f(x)\varphi_n(x)\,dx}{\int_0^L \varphi^2(x)\,dx}$$

then, the remainder of the task is to **solve for the variable coefficient** $b_n(t)$.

3. **Solve for the coefficient** by **attempting to plug back into the PDE**:

$$u_t = ku_{xx} + Q(x,t)$$

first, expand $Q(x,t)$ so we have all terms using $\varphi_n(x)$:

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\varphi_n(x)$$
$$q_n(t) = \frac{\int_0^L Q(x,t)\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx}.$$

then, we need consider:

- u_t, which we **can differentiate term by term** since u_t is continuous

- u_x, which we **can differentiate term-by-term** because the 2L extension of u is **continuous** (since $u(0,t) = u(L,t) = 0$, i.e. we have Dirichlet B.C.)
- u_{xx}, which we **can differential term-by-term** again since now it will be a cosine series.

(if you needed revision, go to subsection 4.3) Therefore, we needed up with:

$$\begin{aligned} u_t &= \sum_{n=1}^{\infty} b_n'(t)\varphi_n(x) \\ u_{xx} &= \sum_{n=1}^{\infty} b_n(t)\varphi_n''(x) = -\sum_{n=1}^{\infty} b_n(t)\lambda_n\varphi_n(x). \end{aligned}$$

since we know that $\varphi'' + \lambda\varphi = 0$. Now, we *plug back to the PDE and obtain*:

$$b_n'(t) + \lambda_n k b_n(t) = q_n(t)$$

this becomes an ODE, and using the **method of integration factors**, we obtain:

$$\begin{aligned} e^{-\lambda_n kt}\left(e^{\lambda_n kt} b_n(t)\right)' &= q_n(t) \\ e^{\lambda_n kt} b_n(t) &= b_n(0) + \int_0^t e^{\lambda_n ks} q_n(s)\, ds \\ b_n(t) &= b_n(0)e^{-\lambda_n kt} + \int_0^t e^{-\lambda_n k(t-s)} q_n(s)\, ds. \end{aligned}$$

and $b_n(0)$ is already computed in step 2, so we have solved the problem

□

8.2.2 Eigenfunction Expansion with Green's Formula

The more **general solution** would be **not using homogeneous B.C.** (i.e. maybe we needed it for some reason), then basically:

- we would not be able to take **term-by-term differentiation** easily
- we **need to take integrals**, i.e. use Green's Formula

Consider the problem we had before:

$$u_t = ku_{xx} + Q(x,t)$$

with now the boundary condition:

$$\begin{cases} u(0,t) = A(t) \\ u(L,t) = B(t) \end{cases}$$

and the initial condition:

$$u(x,0) = f(x)$$

Theorem 8.5: Solutino to Unsteady B.C.

This would be the **most general version** for this type of problem. Consider the setup above, we have:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(x)$$

where $\varphi_n(x)$ is the eigenfunction of the **related homogeneous problem** (i.e. same as above), and that:

$$b_n(t) = b_n(0)e^{-\alpha_n t} + \int_0^t e^{-\alpha_n(t-s)}h_n(s)\,ds$$

for:

- $\alpha_n = \lambda_n k$
- $h_n(t) = h_n(s)$ is all the extra work:

  $$h_n(t) = \frac{k\left[A(t)\varphi'(0) - B(t)\varphi'(L)\right]}{\int_0^L \varphi_n^2(x)\,dx} + q_n(t)$$

  which **collapse to $q_n(t)$ if we had a homogeneous B.C. For** $A(t) = B(t) = 0$. Then we get the same equation as the previous section.

- $\bar{q}_n(t)$ is defined to be:

  $$\bar{Q}(x,t) = \sum_{n=1}^{\infty} \bar{q}_n(t)\varphi_n(x)$$

Proof. The idea is basically the same as the previous section

1. Compute the eigenfunction to the relate PDE. This part is the *same as previous section, hence skipped*

2. **Use Eigenfunction Expansion** to write down the solution:

   $$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(x)$$

   for φ_n of the related homogeneous B.C. version PDE, and then we can compute $b_n(0)$ for later use:

   $$b_n(0) = \frac{\int_0^L f(x)\varphi_n(x)\,dx}{\int_0^L \varphi^2(x)\,dx}$$

3. **Find out the variable coefficient** $b_n(t)$, by the same thought:

   - u_t we can differentiate term-by-term since u_t is assumed piece-wise smooth

- u_x now **cannot differentiate term-by-term**, because the 2L extension of u is not continuous ($A(t), B(t) \neq 0$). i.e. not homogeneous B.C.

Then, we start from the reverse. Let:

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} \gamma_n(t)\varphi_n(x)$$

we would like to **relate this back to** $b_n(t)$ by:

$$\gamma_n(t) = \frac{\int_0^L u_{xx}\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx}$$

(recall that we had done this once before, where we directly computed $\int_0^L u_{xx}\varphi_n(x)\,dx$ using *integration by parts* [see subsection 4.4]) Now, we use a **different technique of Green's Formula**:

$$\int_0^L (u\mathcal{L}[v] - v\mathcal{L}[u])\,dx = p\left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right)\Big|_0^L$$

for $\mathcal{L} \equiv \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$ **being any Sturm-Louisville operator**. In our case, consider:

$$\mathcal{L}[u] = \frac{\partial^2 u}{\partial x^2}$$

with $p = 1, q = 0$. Then we have:

$$\int_0^L (uv_{xx} - vu_{xx})\,dx = (uv_x - vu_x)|_0^L$$

$$\int_0^L vu_{xx}\,dx = \int_0^L uv_{xx}\,dx - (uv_x - vu_x)|_0^L.$$

since we **needed to compute** $\int_0^L u_{xx}\varphi_n(x)\,dx$, we let $v = \varphi_n(x)$, then we get:

$$\begin{aligned}\int_0^L \varphi_n(x)u_{xx}\,dx &= \int_0^L u\varphi_n''\,dx - (u\varphi_n' - \varphi_n u_x)|_0^L \\ &= -\lambda_n\int_0^L u\varphi_n\,dx - u(L,t)\varphi'(L) + u(0,t)\varphi_n'(0) \\ &= -\lambda_n\int_0^L u\varphi_n\,dx - B(t)\varphi_n'(L) + A(t)\varphi_n'(0).\end{aligned}$$

since we knew that $\varphi(x)$ satisfy the homogeneous B.C. of $\varphi(0) = \varphi(L) = 0$. Now, substituting back to γ_n we will see that:

$$\begin{aligned}\gamma_n(t) &= \frac{\int_0^L u_{xx}\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx} \\ &= -\lambda_n\frac{\int_0^L u\varphi_n\,dx}{\int_0^L \varphi_n^2(x)\,dx} + \frac{A(t)\varphi_n'(L) + B(t)\varphi_n'(0)}{\int_0^L \varphi_n^2(x)\,dx} \\ &= -\lambda_n b_n(t) + \frac{A(t)\varphi_n'(L) + B(t)\varphi_n'(0)}{\int_0^L \varphi_n^2(x)\,dx}.\end{aligned}$$

Now, we can **substitute back to the PDE**:

$$\begin{aligned} u_t &= ku_{xx} + Q \\ b_n'(t) + \lambda_n k b_n(t) &= \frac{A(t)\varphi_n'(L) + B(t)\varphi_n'(0)}{\int_0^L \varphi_n^2(x)\,dx} + q_n(t) \\ &= h_n(t). \end{aligned}$$

for:

$$h_n(t) \equiv \frac{A(t)\varphi_n'(L) + B(t)\varphi_n'(0)}{\int_0^L \varphi_n^2(x)\,dx} + q_n(t)$$

then, this becomes the ODE that we can solve via *integrating factor as the previous problem*:

$$b_n(t) = b_n(0)e^{-\alpha_n t} + \int_0^t e^{-\alpha_n(t-s)} h_n(s)\,ds$$

for $\alpha_n = k\lambda_n$, and that:

$$b_n(0) = \frac{2}{L}\int_0^L f(x)\varphi_n(x)\,dx$$

which completes the proof.

$\square$

Note:

Notice that we have solved the PDE with **inhomogenous B.C.**:

$$\begin{aligned} u(0,t) &= A(t) \\ u(L,t) &= B(t). \end{aligned}$$

with **homogeneous eigenfunctions**:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(x)$$

which sounds contradictory because those have $\varphi_n(0) = \varphi_n(L) = 0$ But recall that this is the property of Fourier Series if the **2L extension of u is not continuous**, which we know is the case because the boundary condition is not homogeneous/Dirichlet.

Therefore, technically at those discontinuities, we needed to compute the average for the series (like we did before). This also implies that the functions **converges slowly**, i.e. we need lots of terms to actually get accurate $u(x,t)$.

- on the other hand, approximation if we have a *homogeneous boundary condition (previous section)* would converge faster (i.e. need less terms)

Additionally, it turns out that we could **further simplify the form of the above solution**. We can define:

$$G(x,t;y,s) = \frac{2}{L}\sum_{n=1}^{\infty} \varphi_n(x)\varphi_n(y)e^{-\alpha_n(t-s)}$$

then, this will be useful because if we expand the expression to its actual terms:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(x)$$

by actually putting out expression of $b_n(t)$ in

1. the **source term** would be:

$$\frac{2}{L}\int_0^t \int_0^L \sum_{n=1}^{\infty} Q(y,t)\varphi_n(y)e^{-\alpha_n(t-s)}\varphi_n(x)\,dy\,ds.$$

   for basically expanding the $h_n(x)$ term out and having:

$$q_n(t) = \frac{2}{L}\int_0^L Q(y,t)\varphi_n(y)\,dy$$

   using the G term it becomes:

$$\int_0^t \int_0^L G(x,t;y,s)Q(y,t)\,dy\,dt$$

2. the **initial condition term** was:

$$\frac{2}{L}\sum_{n=1}^{\infty}\int_0^L f(y)\varphi(y)\varphi_n(x)e^{-\alpha_n t}\,dy$$

   and using G:

$$\int_0^L G(x,t;y,0)f(y)\,dy$$

3. the **boundary term** becomes:

$$\int_0^t k\frac{\partial}{\partial y}G(x,t;0,s)\,ds - \int_0^t k\frac{\partial}{\partial y}G(x,t;L,s)B(s)\,ds$$

So in the end, $G(x,t;y,s)$ would have taken into account of all the unsteady terms, including *initial condition, source, and boundary condition*.

8.3 Forced Vibrating Membrane

This is an **example using the above techniques**. In particular, I am going to use the *inhomogeneous boundary condition approach*.

Consider the problem of a **vibrating 2D membrane**:

$$u_{tt} = c^2\nabla^2 u + Q(x,t)$$

which is the *wave equation*, and the $Q(x,t)$ term specifies the **source/forced vibration**.

The B.C. in this case is:

$$u(x,t) = g(x,t), \quad x \in \partial\Omega$$

and the I.C.:

$$\begin{cases} u(x,0) = \alpha(x) \\ \frac{\partial u}{\partial x}(x,0) = \beta(x) \end{cases}$$

Before solving this, we will encounter something below:

Theorem 8.6: Second-Order Linear Nonhomogenous Differential Eq

If we have an equation of the form:

$$\frac{d^2A(t)}{dt^2} + c^2\lambda A = q$$

the **general solution** is a **particular solution plus a linear combination of homogeneous solutions**.

In this case, the homogeneous solution is sine and cosine, so the genreal solution looks like:

$$A(t) = c_1 \cos\left(c\sqrt{\lambda}t\right) + c_2 \sin\left(c\sqrt{\lambda}t\right) + \text{particular}$$

Theorem 8.7: Solution to Forced Vibrating Membrane

The solution is:

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\varphi_n(x)$$

where $\varphi_n(x)$ *comes from Helmholtz equation*, and can be solved if we specify the geometry.

and that:

$$A_n(t) = C_n \cos\left(c\sqrt{\lambda_n}t\right) + D_n \sin\left(c\sqrt{\lambda}t\right) + \text{particular}$$

with:

$$\text{particular} = \frac{1}{\omega_n}\int_0^t \sin\left(\omega_n(t-\tau)\right)s_n(\tau)\,d\tau$$

where:

- $s_n(t)$ contains *both source and boundary*

$$s_n(t) \equiv q_n(t) - c^2 r_n(t)$$

- the boundary term is in fact:

$$r_n(t) = \frac{\int_{\partial\Omega} g(x,t) \frac{\partial \varphi_n}{\partial n} \, dS}{\int_\Omega \varphi_n^2(x) \, dV}$$

this **solved $A_n(t)$**, and together with C_n, D_n obtained from initial condition:

$$\begin{cases} A_n(0) = C_n \\ A_n'(0) = \omega_n D_n \end{cases}$$

Proof. So, we just the routine established in the previous section.

1. Start with a **homogeneous problem and find the eigenfunctions**. In this case, we have the **Helmholtz Equation**:

   $$\nabla^2 \varphi + \lambda \varphi = 0, \quad x \in \Omega$$

   and that:

   $$\varphi(x) = 0, \quad x \in \partial\Omega$$

   recall that Helmholtz Equation had the property of:

   (a) φ_n is the **complete set**

   (b) $\lambda_n > 0$ if Dirichlet B.C. (which we have)

   However, the specific eigenfunctions depend on the **geometric shape of the region**. Explicit formulas can be obtained only for certain relatively simple geometries. (e.g. subsection 7.2)

2. Then, **expand the solution in eigenfunctions**:

   $$u(x,t) = \sum_{n=1}^{\infty} A_n(t) \varphi_n(x)$$

   again, we can immediately compute $A_n(0), A_n'(0)$ from initial condition here.

   Additionally, let us expand $Q(x,t)$ now:

   $$Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \varphi_n(x)$$

   and that:

   $$q_n(t) = \frac{\int_\Omega \varphi_n(x) Q(x,t) \, dV}{\int_\Omega \varphi_n^2(x) \, dV}$$

3. Now, we need to **compute the coefficients**.

   (a) the time derivatives are always assumed to be continuous, so we can *take term-by-term differentiation*:

   $$u_{tt} = \sum_n A_n''(t) \varphi_n(x)$$

(b) the term $\nabla^2 u$, which we want to avoid since the *geometry is complex at boundary*.

Therefore, we do the following approach:

$$u_{tt} = \sum_n A_n''(t)\varphi_n(x) = c^2\nabla^2 u + Q$$

$$A_n''(t) = \frac{1}{\int_\Omega \varphi_n^2(x)\,dV} \int_\Omega \left(c^2\varphi_n\nabla^2 u + \varphi_n Q\right)\,dV.$$

which becomes integrals, and computing the terms separately:

(a) consider:

$$c^2 \int_\Omega \varphi_n(x)\nabla^2 u\,dx.$$

using **Green's Formula**:

$$\begin{aligned}\int_\Omega \left(v\nabla^2 u - u\nabla^2 v\right)\,dV &= \int_{\partial\Omega} (v\nabla u - u\nabla v)\cdot\hat{n}\,dS \\ &= \int_{\partial\Omega}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)\,dS.\end{aligned}$$

Then we have:

$$\begin{aligned}\int_\Omega \varphi_n\nabla^2 u\,dV &= \int_\Omega u\nabla^2\varphi_n\,dV + \int_{\partial\Omega}\left(\varphi_n\frac{\partial u}{\partial n} - u\frac{\partial\varphi_n}{\partial n}\right)\,dS \\ &= -\lambda_n\int_\Omega u\varphi_n\,dV - \int_{\partial\Omega} g(x,t)\frac{\partial\varphi_n}{\partial n}\,dS.\end{aligned}$$

since we know that $\varphi_n(x) = 0, x \in \partial\Omega$ and the boundary condition. Notice that **if we have homogeneous boundary condition, then the $g(x,t)$ would have disappeared**.

Last but not least, combining with the factor:

$$\begin{aligned}\frac{\int_\Omega \varphi_n\nabla^2 u\,dV}{\int_\Omega \varphi_n^2(x)\,dV} &= -\lambda_n\frac{\int_\Omega u\varphi_n\,dV}{\int_\Omega\varphi_n^2(x)\,dV} - \frac{\int_{\partial\Omega} g(x,t)\frac{\partial\varphi_n}{\partial n}\,dS}{\int_\Omega\varphi_n^2(x)\,dV} \\ &= -\lambda_n A_n(t) - r_n(t).\end{aligned}$$

where we defined $r_b(t)$to basically contain the **boundary term**:

$$r_n(t) = \frac{\int_{\partial\Omega} g(x,t)\frac{\partial\varphi_n}{\partial n}\,dS}{\int_\Omega\varphi_n^2(x)\,dV}$$

(b) the other source term is actually simple:

$$\frac{\int_\Omega Q(x,t)\varphi_n(x)\,dV}{\int_\Omega\varphi_n^2(x)\,dV} = q_n(t)$$

because we had:

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\varphi_n(x)$$

now, **putting everything back**:

$$\begin{aligned}A_n''(t) - q_n(t) &= -c^2\lambda_n A_n(t) - c^2 r_n(t)\\ A_n''(t) + c^2\lambda_n A_n(t) &= q_n(t) - c^2 r_n(t)\\ A_n''(t) + \omega_n^2 A_n(t) &= s_n(t).\end{aligned}$$

where we have defined:

- $\omega_n = c\sqrt{\lambda_n}$
- the **source and boundary combined** to be:

$$s_n(t) \equiv q_n(t) - c^2 r_n(t)$$

However, to solve the below equation:

$$A_n''(t) + \omega_n^2 A_n(t) = s_n(t)$$

we needed the method of *particular solution*, so that:

$$A_n(t) = C_n \cos\left(c\sqrt{\lambda_n}t\right) + D_n \sin\left(c\sqrt{\lambda}t\right) + \text{particular}$$

to solve for the particular solution, we can use the **method of undetermined coefficients**:

$$\text{let particular} = a_n(t)\cos(\omega_n t) + b_n(t)\sin(\omega_n t)$$

note that if $a_n(t) = a_n, b_n(t) = b_n$ *becomes constant*, then we would get $s_n(t) = 0$.
It turns out the solution is:

$$\text{particular} = \frac{1}{\omega_n}\int_0^t \sin(\omega_n(t-\tau))s_n(\tau)\,d\tau$$

this **solved $A_n(t)$**, and together with C_n, D_n obtained from initial condition:

$$\begin{cases} A_n(0) = C_n \\ A_n'(0) = \omega_n D_n \end{cases}$$

□

***Example*: Resonance**

Consider the **special case of**:

$$s_n(t) = q_n(t) - 0 = \gamma_n \cos(\omega t)$$

where basically, we have *no boundary term* but a *source term of $\gamma_n \cos(\omega t)$*. This is also called a **sinusoidal forcing**.

Then we have:

$$A_n''(t) + \omega_n A_n(t) = \gamma_n \cos(\omega t)$$

and notice that ω for the source has *nothing to do with the ω_n*. Then:

$$A_n(t) = C_n \cos\left(c\sqrt{\lambda_n}t\right) + D_n \sin\left(c\sqrt{\lambda}t\right) + \text{particular}$$

and in this case, it **can be guess that**:

$$\text{particular} = F_n \cos(\omega t)$$

so that it will satisfy the ODE. This means that:

$$\begin{aligned}\left(-\omega^2 F + \omega_n^2 F\right)\cos(\omega t) &= \gamma_n \cos(\omega t)\\ F_n &= \frac{\gamma_n}{\omega_n^2 - \omega^2}.\end{aligned}$$

for $\omega \neq \omega_n$, which means that:

$$\text{particular} = \frac{\gamma_n}{\omega_n^2 - \omega^2}\cos(\omega t)$$

and notice that:

- $\omega_n = c\sqrt{\lambda_n}$ are the **natural frequencies**
- forced response (particular solution) has the *same frequency as the source*
- amplitude of the forced response **becomes large when near natural frequency**

now, for $\omega = \omega_n$, we need to **solve the particular solution again from the ODE**, and we will get:

$$\text{particular} = \frac{\gamma_n}{\omega} t \sin(\omega t)$$

where basically $\omega = \omega_n$ makes this thing **never stop/grow linear in time $\iff$ resonance**

1. since this forced response grows in time, eventually the *object might break*
2. or, our assumption/PDE of wave equation might no longer be valid
   - assumption of small displacement
   - ignorance of *damping terms*

9 Green's Function

Before talkie about the theory, let's **review some Linear Algebra** related topics so that you could see some *analogies later*.

Theorem 9.1: Symmetrical Matrix Decomposition

If a square matrix A of size $N \times N$ is **symmetric**, then we know the following:

- decomposition

$$A = V\Lambda V^{-1} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & ... \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & ... \\ 0 & \lambda_2 & ... \\ 0 & 0 & ... \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ ... \end{bmatrix}$$

  where $\vec{v}_n, \lambda_n$ is the n-th eigenvector/eigenvalue
  Alternatively, this can be written as:

$$A = \sum_{n=1}^{N} \lambda_n \vec{v}_n \vec{v}_n^T$$

- the inverse can be computed easily:

$$A^{-1} = \sum_{n=1}^{N} \frac{1}{\lambda_n} \vec{v}_n \vec{v}_n^T$$

9.1 Introduction with ODE

Consider an ODE with:

$$\mathcal{L}[u(x)] = f(x)$$

where I have $\mathcal{L}[u] = \frac{d}{dx}\left(p\frac{d}{dx}u\right) + q(x)u(x)$ being a *Sturm-Louisville operator*, and we have B.C.:

$$u(a) = u(b) = 0$$

Reminder:

Since it is a ***S-L problem****, we should remember that it:*

1. *has a* ***complete set of orthogonal eigenfunctions*** φ_n*:*

$$L\left[\varphi_n\right] + \lambda_n\varphi_n = 0$$

for $\varphi_n(a) = \varphi_n(b) = 0$ *satisfying the B.C.*

Theorem 9.2: Green's Function for Simple ODE

For the above problem, we can **also have** the solution written in Green's Formula:

$$u(x) = \sum_{n=1}^{\infty} a_n\varphi_n = \int_a^b G(x,y)f(y)\,dy$$

where the eigenfunction is computed from

$$L\left[\varphi_n\right] + \lambda_n\varphi_n = 0$$

with the **Green's Function for this problem** being:

$$G(x,y) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\frac{\varphi_n(x)\varphi_n(y)}{\int_a^b \varphi_n^2(z)\,dz}$$

essentially we do this because we want to have an *attempt to undone the S-L operator* like in Linear Algebra:

$$\begin{aligned} A\vec{x} &= \vec{b} \\ \vec{x} &= A^{-1}\vec{b}. \end{aligned}$$

and we have here:

$$\begin{aligned} -\mathcal{L}[u] &= f(x) \\ u &= \int_a^b G(x,y)f(y)\,dy. \end{aligned}$$

Proof. First, consider an eigenfunction expansion:

$$u(x) = \sum_{n=1}^{\infty} a_n\varphi_n(x)$$

from:

$$L[\varphi_n] + \lambda_n \varphi_n = 0$$

where we can *compute coefficients from orthogonality*:

$$a_n = \frac{\int_a^b u(x)\varphi_n(x)\,dx}{\int_a^b \varphi^2(x)\,dx}$$

which technically solved the problem. However, suppose we want to *derive the solution by undo-ing the S-L operation*:

$$-\mathcal{L}[u] = f(x)$$

so that we get:

$$u(x) = \text{something}(f(x))$$

Consider the integral $\int_a^b u\varphi_n\,dx$, using Green's Formula:

$$\int_a^b \left(v\mathcal{L}[u] - v\mathcal{L}[u]\right)\,dx = 0$$

because the B.C. is homogeneous. Then, letting $v = \varphi_n$, we get:

$$\begin{aligned}
\int_a^b \left(\varphi_n\mathcal{L}[u] - u\mathcal{L}[\varphi_n]\right)\,dx = 0 \\
\int_a^b \varphi_n(x)f(x)\,dx &= \int_a^b \lambda_n(x)\varphi_n(x)u(x)\,dx \\
\frac{\int_a^b \varphi_n(x)f(x)\,dx}{\int_a^b \varphi_n^2\,dx} &= \lambda_n a_n[u] \\
a_n[f] &= \lambda_n a_n[u] \\
a_n[u] &= \frac{1}{\lambda_n}a_n[f].
\end{aligned}$$

Therefore, I get:

$$\begin{aligned}
u(x) = \sum_{n=1}^{\infty} a_n[u]\varphi_n(x) &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} a_n[f]\varphi_n(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \frac{\int_a^b f(y)\varphi_n(y)\,dy}{\int_a^b \varphi_n^2(y)\,dy}\varphi_n(x) \\
&= \int_a^b f(y)\left[\sum_{n=1}^{\infty} \frac{1}{\lambda_n}\frac{\varphi_n(x)\varphi_n(y)}{\int_a^b \varphi_n^2(z)\,dz}\right]\,dy.
\end{aligned}$$

And now we have **obtained the Green's Function for this ODE**, in that:

$$G(x,y) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\frac{\varphi_n(x)\varphi_n(y)}{\int_a^b \varphi_n^2(z)\,dz}$$

so that we have:

$$u(x) = \int_a^b G(x,y)f(y)\,dy$$

where we have:

1. this Green's Function looks similar to the linear algebra **inverse**:

$$A^{-1} = \sum_{n=1}^{N} \frac{1}{\lambda_n} \vec{v}_n \vec{v}_n^T$$

   and that:

$$G(x, y) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \frac{\varphi_n(x)\varphi_n(y)}{\int_a^b \varphi_n^2(z)\, dz}$$

   which looks *even more alike if φ_n is normalized so that $\int_a^b \varphi_n(z)\, dz = 1$*

2. *undone the S-L operator*, like in Linear Algebra, we have:

$$\begin{aligned} A\vec{x} &= \vec{b} \\ \vec{x} &= A^{-1}\vec{b}. \end{aligned}$$

   and we have here:

$$\begin{aligned} -\mathcal{L}[u] &= f(x) \\ u &= \int_a^b G(x, y) f(y)\, dy. \end{aligned}$$

□

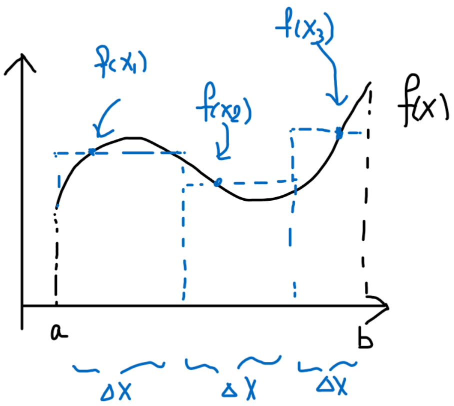
Note:

The Green function for this ODE:

- **exists** $\iff \lambda_n \neq 0$ (otherwise we cannot use $\frac{1}{\lambda_n}$)
- it is **symmetric** in that $G(x, y) = G(y, x)$

9.2 Computing Green's Function for ODEs

However, $G(x, y)$ is hard to calculate now. So we can attempt to use **Dirac Delta Function** to represent an **arbitrary function** $f(x)$



Starting from Dirac Delta Functions as Square Pulses of *height* $\frac{1}{\Delta x}$ with *width* Δx , we can write down:

$$f(x) \approx \sum_i f(x_i)\delta_{\Delta x}(x - x_i)\Delta x$$

in the limit, we have:

$$f(x) = \int_a^b f(y)\delta(x - y)\, dy$$

where also notice that:

- delta function is *also symmetric*

$$\delta(x - y) = \delta(y - x)$$

- the **anti-derivative of a delta function** $\delta(x)$ is a **Heaviside Function** $H(x)$:

$$\frac{d}{dx}H(x) = \delta(x)$$

Theorem 9.3: Greens' Function

Consider the ODE:

$$\mathcal{L}[u(x)] = \frac{d^2u}{dx^2} = f(x)$$

with *homogeneous B.C.*. Then From theorem 9.2, we know that we are dealing with:

$$u(x) = \sum_{n=1}^{\infty} a_n\varphi_n = \int_a^b G(x, y)f(y)\, dy$$

Let $f(x) = \delta(x - x_0)$, then we can obtain *ODEs for* $G(x, y)$, and the solution in this cases looks like:

$$G(x,y) = \begin{cases} \left(\frac{y-L}{L}\right) x, & 0 < x < y \\ \left(\frac{x-L}{L}\right) y, & y < x < L \end{cases}$$

which is **symmetric in x, y**

Proof. The problem is:

$$\mathcal{L}[u(x)] = \frac{d^2 u}{dx^2} = f(x)$$

The aim is to find out $G(x,y)$ in the below equation

$$u(x) = \sum_{n=1}^{\infty} a_n \varphi_n = \int_a^b G(x,y) f(y)\, dy$$

1. **A trick is to** let $f(x) = \delta(x - x_0)$, then we get:

$$u(x) = \int_a^b \delta\,(y - x_0)\, G(x,y)\, dy = G(x, x_0)$$

Note:

This trick is justified because we know that the **general source term is $f(x)$**:

$$f(x) = \int f(y) \delta(x-y)\, dy$$

and suppose we have a *singular source, where* :

$$\mathcal{L}[u] = f(y)\delta(x-y)$$

which means the source is *just at one point $x = y$*, then the solution is:

$$u = f(y) G(x,y)$$

since basically we are going to solve $G(x,y)$ for:

$$\mathcal{L}[G] = \delta(x-y)$$

Therefore, it **can be justified then**:

$$u = \int_0^L f(y) G(x,y)\, dy, \quad \mathcal{L}[u] = f(x)$$

since then we are just **summing all contributions of G, which comes from a singular source**.

Then, using the trick, we know *something more*:

$$\mathcal{L}[G(x, x_0)] = f(x) = \delta(x - x_0)$$
$$G(0,y) = G(L,y) = 0.$$

Then, the problem becomes essentially solving an ODE of $G(x, x_0)$, with $\mathcal{L} = \frac{d^2}{dx^2}$.

2. To solve the above ODE that **has a delta function**, there are two ways:

   - Method 1: Using **Integrals**. We know that:

     $$\frac{d^2}{dx^2}G(x, x_0) = \frac{d^2}{dx^2}G(x, y) = \delta(x - y)$$

     Then integrating gives:

     $$G'(x, y) = H(x - y) + \alpha$$

     Doing it again, we get:

     $$G(x, y) = \int_0^x H(z - y)\, dz + \alpha x + \beta$$

     However, we know that the Heaviside function would behave like:

     $$H(z - y) = \begin{cases} 0 & z < y \\ 1 & z > y \end{cases}$$

     Therefore, this means that the *integral can be evaluated*:

     $$\int_0^x H(z - y)\, dz = \begin{cases} 0 & x < y \\ x - y & x > y \end{cases}$$

     Now, we can use the **two boundary conditions to solve for the two coefficients**:

     $$G(0, y) = \beta = 0.$$

     and, since $y \leq L$, since the big picture was doing $\int_0^L G(x, y) f(y)\, dy$, so we have:

     $$G(L, y) = (L - y) + \alpha L = 0$$
     $$\alpha = \frac{y - L}{L}.$$

     Now, *putting everything back*, we basically have solved it:

     $$G(x, y) = \begin{cases} 0 + \frac{y-L}{L}x = \frac{y-L}{L}x, & x < y \\ x - y + \frac{y-L}{L} = \frac{x-L}{L}y, & x > y \end{cases}$$

     and **notice that $G(x, y)$ is symmetric**

   - Method 2: Solving **per Region** Since the ODE we are dealing with involves a delta function, we can separate the ODE into *two regions like we did for Quantum Mechanics*:

     $$\frac{d^2}{dx^2}G(x, y) = \delta(x - y)$$

     then into two parts:

     $$\begin{cases} G''(x, y) = 0, & 0 < x < y \\ G''(x, y) = 0, & y < x < L \end{cases}$$

Then we just need to solve the two simple ODEs, which we just get:

$$\begin{cases} G_I(x,y) = ax + b, & 0 < x < y \\ G_{II}(x,y) = cx + d, & y < x < L \end{cases}$$

Though it appears that we have *four unknown coefficients but only two boundary conditions*, we have some extra constraints in this problem, **namely continuity and jump continuity**.

First, using the two homogeneous boundary condition, I get:

$$\begin{cases} G_I(x,y) = ax, & 0 < x < y \\ G_{II}(x,y) = cx - cL, & y < x < L \end{cases}$$

using $G(0,y) = 0$ and $G(L,y) = 0$ respectively. Afterwards, we can **first use the jump continuity** due to the presence of the delta function:

$$\begin{aligned} G''(x,y) &= \delta(x-y) \\ \int_{y-\epsilon}^{y+\epsilon} G''(x,y)\,dy &= \int_{y-\epsilon}^{y+\epsilon} \delta(x-y)\,dy \\ G'(x,y)\big|_{y-\epsilon}^{y+\epsilon} &= 1 \\ G'_{II}(y+\epsilon,y) - G'_I(y-\epsilon,y) &= 1 \\ G'_{II}\left(y^+,y\right) - G'_I(y^-,y) &= 1. \end{aligned}$$

which make sense because:

– we knew in Method 1 that $G'(x,y) = H(x-y)$, so there *is a jump at $x = y$*

Now, *computing the derivates and putting them in*, I get:

$$\begin{aligned} c - a &= 1 \\ a &= c - 1. \end{aligned}$$

Lastly, use the **"normal" continuity equation**:

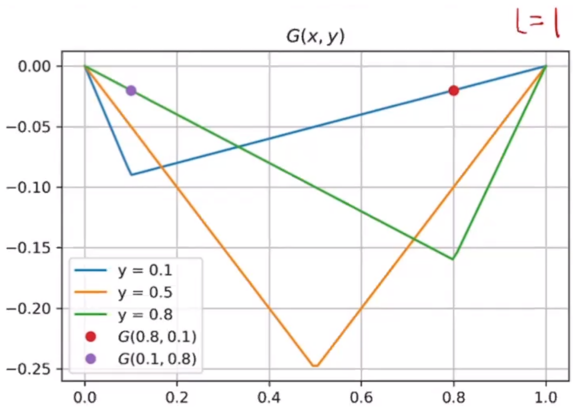
$$\begin{aligned} G_I(y,y) &= G_{II}(y,y) \\ (c-1)y &= c(y-L) \\ y &= cL \\ c &= \frac{y}{L}. \end{aligned}$$

Now we got all the coefficients, putting them back into the Green's Function and we see:

$$G(x,y) = \begin{cases} \left(\frac{y}{L} - 1\right)x = \left(\frac{y-L}{L}\right)x, & 0 < x < y \\ \frac{y}{L}(x-L) = \left(\frac{x-L}{L}\right)y, & y < x < L \end{cases}$$

Regardless of which method, we have now found the Green's Function. □

Graph of the above Green's Function:



Interpretation for Green's Function:

Because we know that:

$$\mathcal{L}\,[G] = \delta(x - x_0)$$

this means that $G(x, y)$ - the solution - is the **response** as a function of x to the $\delta(x - x_0) = f(x)$ - the source - **as a source at** x_0.
Therefore, the equation:

$$u(x) = \int_a^b f(x_s) G(x, x_s)\, dx$$

can be seen as $u(x)$ being the *superposition of all $G(x, x_s)$, which is a response to a source at x_0*.

Note:

Notice that we solved the ODEs quite easily *in part because the boundary condition was homogeneous*. If we have an in-homogeneous B.C., you will see that there are some interesting ways to deal with it.

- one obvious way that we had before was to *subtract from it a reference solution*, which would work but not discussed next.
- the other more **common way**, which is discussed next, is to solve the non-homogeneous problem with the **easier, homogeneous B.C. version Green's Function G plus a trailing term** that solves the non-homogeneous B.C.

9.2.1 Symmetry of Green's Function

Corollary 9.1: Symmetry of Green's Function

In general, **Green's Function is symmetric** in that:

$$G(x, y) = G(y, x)$$

if we have a **self-adjoint/hermitian operator** for the ODE.

Proof. Suppose we have a self-adjoint operator $\mathcal{L}$. We already know:

$$\mathcal{L}\left[G(x, y)\right] = \delta(x - y)$$

since $\mathcal{L}$ is self-adjoint (*e.g. a S-L operator with homogeneous B.C.*), then consider:

$$\int_a^b v\mathcal{L}[u]\, dx = \int_a^b u\mathcal{L}[v]\, dx$$

letting $u = G(x, y), v = G(x, z)$, we have:

$$\begin{aligned}
\int_a^b G(x, z)\mathcal{L}[G(x, y)]\, dx &= \int_a^b G(x, y)\mathcal{L}[G(x, z)]\, dx \\
\int_a^b G(x, z)\delta(x - y)\, dx &= \int_a^b G(x, y)\delta(x - z)\, dx \\
G(y, z) &= G(z, y).
\end{aligned}$$

This finishes the proof. $\square$

9.2.2 Non-Homogeneous Boundary Condition

Before, we solved the Green's function having the **B.C. for PDE being homogeneous**.

Reminder:

> *The previous case was:*
>
> $$\mathcal{L}[u(x)] = \frac{d^2u}{dx^2} = f(x)$$
>
> *with* homogeneous B.C.. *So we ended up solving* G *with* ***homogeneous B.C.****:*
>
> $$\begin{aligned} \frac{d^2}{dx^2}G(x,y) &= \delta(x-y) \\ G(0,y) &= G(L,y) = 0. \end{aligned}$$
>
> *Then, by letting* $f(x) = \delta(x - x_0)$*, we can obtain* ODEs for $G(x,y)$*, and the solution in this cases looks like:*
>
> $$G(x,y) = \begin{cases} \left(\frac{y-L}{L}\right) x, & 0 < x < y \\ \left(\frac{x-L}{L}\right) y, & y < x < L \end{cases}$$
>
> *which is* ***symmetric in*** x, y

Now, we consider the general non-homogeneous B.C. with:

$$\begin{aligned} u(0) &= \alpha \\ u(L) &= \beta. \end{aligned}$$

And the general idea would be to **use the $G(x,y)$ of the homogeneous B.C. version to solve non-homogeneous case**

- alike the approach of *solving non-homogeneous problem using related homogeneous eigenfunction* φ_n.
- in fact, one big reason is that G is essentially **composed of homogeneous B.C. eigenfunctions**

Theorem 9.4: Solution to Non-homogeneous ODE Problem

Consider the setup of the *same ODE*:

$$\frac{d^2}{dx^2}u = f(x)$$

but with **non-homogeneous B.C.**:

$$\begin{aligned} u(0) &= \alpha \\ u(L) &= \beta. \end{aligned}$$

The idea is to **use the $G(x, y)$ of the related homogeneous version of the problem**, so that we have:

$$u(y) = \int_0^L G(x, y)f(x)\,dx + \beta\frac{y}{L} - \alpha\frac{y - L}{L}.$$

where we see that:

- $G(x, y)$ is the homogeneous version:

$$\int_0^L (u\mathcal{L}[v] - v\mathcal{L}[u])\,dx = (1)\,(uv' - vu')|_0^L\,.$$

- the term:

$$\int_0^L G(x, y)f(x)\,dx$$

  is satisfying the *source term $f(x)$* of the PDE but was for *homogeneous B.C. Term*

- the other term:

$$\beta\frac{y}{L} - \alpha\frac{y - L}{L}$$

  solves the *boundary term* of the PDE, but was for *homogenous source term*, i.e. $\mathcal{L}[u] = 0$.

Combining them hence makes the perfect sense. (i.e. the analogy of having a reference solution)

Proof. The steps are basically the same as the *eigenfunction expansion* approach, so that:

1. **find out** the related homogeneous version of the PDE, and its $G(x, y)$ (which would hence also have a homogeneous B.C., so easier to solve). In this case, we have already done it, we have

$$G(x, y) = \begin{cases} \left(\frac{y-L}{L}\right)x, & 0 < x < y \\ \left(\frac{x-L}{L}\right)y, & y < x < L \end{cases}$$

2. Now, the trick is to **consider the Green's Formula** with now homogeneous B.C. terms:

$$\int_0^L (u\mathcal{L}[v] - v\mathcal{L}[u])\,dx = (1)\,(uv' - vu')|_0^L\,.$$

   Since we don't have homogeneous B.C. for u, but we *do have homogeneous B.C. for $G(x, y)$*, and since we knew $G''(x, y) = \delta(x - y)$, we have:

$$\begin{aligned} \int_0^L [u(x)\delta(x - y) - G(x, y)f(x)]\,dx &= u(x)G'|_0^L \\ u(y) - \int_0^L G(x, y)\,dx &= u(L)\frac{y}{L} - u(0)\frac{y - L}{L} \\ u(y) &= \int_0^L G(x, y)f(x)\,dx + \beta\frac{y}{L} - \alpha\frac{y - L}{L}. \end{aligned}$$

where we see that:

- the term:
$$\int_0^L G(x,y)f(x)\,dx$$
is satisfying the *source term $f(x)$* of the PDE but was for *homogeneous B.C. Term*

  Notice that this is also one justification why taking $f(x) = \delta(x-y)$ is valid for solving $H(x,y)$ in the *homogeneous case*.
- the other term:
$$\beta\frac{y}{L} - \alpha\frac{y-L}{L}$$
solves the *boundary term* of the PDE, but was for *homogenous source term*, i.e. $\mathcal{L}[u] = 0$.

Combining them hence makes the perfect sense. (i.e. the analogy of having a reference solution)

$\square$

9.2.3 Higher Dimensions with Eigenfunction Expansion

Now, consider the ODE problem at a *higher dimension*, so that we consider a **general domain** Ω with **general boundary** $\partial\Omega$

Note:

This could also be used for solving a **general S-L problem**:

$$\mathcal{L}[u] = f(x)$$

Then, the general steps would be:

1. find the *related homogeneous problem and its eigenfunction*
2. use *eigenfunction expansion* to obtain the solution, and **convert it into a series of Green's Function**
3. End up with an **ODE for Green's Function**, and solve it

The example that I will cover here would be using ODE:

$$\nabla^2 u = f(x), \quad x \in \Omega$$

with first a *homogeneous B.C.*:

$$u(0) = u(L) = 0$$

Note:

If the problem is non-homogeneous in B.C., you can use the approach covered in subsubsection 9.2.2.

Theorem 9.5: Green's Function of Laplacian in Infinite Space

Consider the **ODE for** G, so that:

$$\nabla^2_{(\vec{x})} G\left(\vec{x}, \vec{y}\right) = \delta\left(\vec{x} - \vec{y}\right), \qquad \vec{x} \in \Omega$$

but now, we are having $\vec{x}$ for **infinite space**, so that we **don't have a boundary**. Then, we can utilize some *symmetry property*:

- translational symmetry
- rotational symmetry

so that we could eventually imagine **shifting the source to the origin**, and obtain the solution in **2D**:

$$\begin{aligned} G(\vec{x}, \vec{y}) &= \frac{1}{2\pi} \ln\left(r\right) + \beta \\ &= \frac{1}{2\pi} \ln\left(|\vec{x} - \vec{y}|\right) + \beta. \end{aligned}$$

and in **3D**, it becomes:

$$\begin{aligned} G(\vec{x}, \vec{y}) &= -\frac{1}{4\pi r} + \beta \\ &= -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{y}|} + \beta. \end{aligned}$$

all for $r = |\vec{x} - \vec{y}| > 0$.

Proof. Consider the **ODE for G**, so that:

$$\nabla^2_{(\vec{x})} G(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}), \qquad \vec{x} \in \Omega$$

for $\vec{x}$ in infinite space, i.e. no boundary, then we have:

- **translational symmetry**, so that:

$$G(\vec{x}, \vec{y}) = G(\vec{x} - \vec{y}, 0)$$

- **rotational symmetry**, so that letting (**assuming we are in 2D for this example**):

$$r = |\vec{x} - \vec{y}|$$

  being the **displacement form source** (i.e. source becomes the origin), then we have $G(r, \theta)$ with the ODE becoming:

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial G}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 G}{\partial \theta^2} = \delta(\vec{x} - \vec{y})$$

  and because of the rotational symmetry (as we are in infinite space), $G(r, \theta) = G(r)$, so that:

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial G}{\partial r} \right) = \delta(\vec{x} - \vec{y})$$

Now even if we don't have B.C., we can still solve it so that *for $r > 0$*, i.e. when we are away from source:

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial G}{\partial r} \right) &= 0 \\ rG' &= \alpha \\ G' &= \frac{\alpha}{r} \\ G &= \alpha \ln(r) + \beta. \end{aligned}$$

where $\beta = G(0)$ would be unknown, yet α is solvable by considering a *small ball* *with radius R at the origin*:

$$\begin{aligned}\int_{\text{Ball}_R} \nabla^2 G \, dV_{(\vec{x})} &= \int_{\text{Ball}_R} \nabla \cdot (\nabla G) \, dV_{(\vec{x})} \\ &= \int_{\partial\text{Ball}_R} \nabla G \cdot \hat{n} \, R d\theta \\ &= \int_{\partial\text{Ball}_R} \nabla G \cdot \hat{r} \, R d\theta \\ &= \int_{\partial\text{Ball}_R} \frac{\partial G}{\partial \hat{r}} \, R d\theta \\ &= \int_{\partial\text{Ball}_R} \frac{\alpha}{R} \, R d\theta \\ &= 2\pi\alpha.\end{aligned}$$

However, we also know that:

$$\nabla^2 G = \delta\left(\vec{x} - \vec{y}\right)$$

therefore:

$$1 = 2\pi\alpha, \qquad \alpha = \frac{1}{2\pi}$$

So that the Green's Function for *infinite space* in **2D** is:

$$\begin{aligned}G(\vec{x}, \vec{y}) &= \frac{1}{2\pi} \ln\left(r\right) + \beta \\ &= \frac{1}{2\pi} \ln\left(|\vec{x} - \vec{y}|\right) + \beta.\end{aligned}$$

□

Note:

The version of Green's function in **3D** would be solved analogously, you will basically just use:

- 3D spherical Laplacian
- same symmetry arguement

and end up with:

$$G(\vec{x}, \vec{y}) = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{y}|} + \beta$$

which looks like the Coulomb's Electrostatic Potential with the $\frac{1}{r}$ dependence.

Theorem 9.6: Green's Function of the Higher Dimension ODE

The ODE was:

$$\nabla^2 u = f(x), \quad x \in \Omega$$

with a *homogeneous B.C.*:

$$u(0) = u(L) = 0$$

The *related homogeneous problem*

$$\nabla^2 \varphi_n + \lambda_n \varphi_n = 0$$

being the eigen-equation, and realize that this is the **Helmholtz Equation**.

Then, we will then end up with the Green's Function in Higher Dimension being:

$$\begin{aligned} u(\vec{x}) &= \sum_{n=1}^{\infty} A_n[u]\varphi_n(\vec{x}) \\ &= \int_\Omega \left[\sum_{n=1}^{\infty} -\frac{1}{\lambda_n} \frac{\varphi_n(\vec{x})\varphi_n(\vec{y})}{\int_\Omega \varphi_n^2(\vec{z})\, dV_{(\vec{z})}} \right] f(\vec{y})\, dV_{(\vec{y})} \\ &= \int_\Omega G(\vec{x}, \vec{y}) f(\vec{y})\, dV_{(\vec{y})}. \end{aligned}$$

Lastly, we will use the *Green's Function in the related infinite space problem* to solve: TODO.

Proof. As mentioned above, we will use the *approach of eigenfunction expansion*.

1. firstly, find a *related homogeneous problem*

   $$\nabla^2 \varphi_n + \lambda_n \varphi_n = 0$$

   being the eigen-equation, and realize that this is the **Helmholtz Equation**, so that:

   - φ_n is a complete set

   Before going to the *eigenfunction expansion*, we can compute some useful quantity. Consider:

   $$u = u(\vec{x}), \quad v = \varphi_n(\vec{x})$$

   in 2D or 3D. Then, consider the *Green's Forumla*

   $$\int_\Omega \left(\varphi_n \nabla^2 u - u \nabla^2 \varphi_n\right) dV_{(\vec{x})} = 0.$$

   because we have a **homogeneous B.C.**. Using the fact that $\nabla^2 \varphi_n = -\lambda \varphi_n$, and that $\nabla^2 u = f(x)$ being the ODE, we have:

   $$\int_\Omega u \varphi_n \, dV_{(\vec{x})} = -\frac{1}{\lambda_n} \int_\Omega \varphi_n f(\vec{x})\, dV_{(\vec{x})}$$

   which is *almost the coefficient for the eigenfunction expansion*.

2. next, we do the **eigenfunction expansion**, so that:

$$\begin{aligned}
u(\vec{x}) &= \sum_{n=1}^{\infty} A_n[u]\varphi_n(\vec{x}) \\
&= \sum_{n=1}^{\infty} \left(-\frac{1}{\lambda_n} \frac{\int_\Omega \varphi_n(y) f(y)\, dV_{(y)}}{\int_\Omega \varphi_n^2(z)\, dV_{(z)}} \right) \varphi_n(\vec{x}) \\
&= \int_\Omega \left[\sum_{n=1}^{\infty} -\frac{1}{\lambda_n} \frac{\varphi_n(\vec{x})\varphi_n(\vec{y})}{\int_\Omega \varphi_n^2(\vec{z})\, dV_{(\vec{z})}} \right] f\left(\vec{y}\right)\, dV_{(\vec{y})} \\
&= \int_\Omega G(\vec{x},\vec{y}) f(\vec{y})\, dV_{(\vec{y})}.
\end{aligned}$$

which proved the **validity and the form of the Green's Function in this problem**.

3. once we get the equation:

$$u(\vec{x}) = \int_\Omega G\left(\vec{x},\vec{y}\right) f(\vec{y})\, dV_{(\vec{y})}$$

the rest is the same idea for solving G. The trick is to let $f(\vec{y}) = \delta(\vec{y} - \vec{z})$, so that:

$$u(\vec{x}) = G(\vec{x},\vec{z})$$

the argument that the *generality of source term $f(x)$ of ODE is still preserved* is discussed in previous sections.
Then, we obtain the **ODE for G**, so that:

$$\nabla^2_{(\vec{x})} G\left(\vec{x},\vec{z}\right) = \delta\left(\vec{x} - \vec{z}\right), \qquad \vec{x} \in \Omega$$

and the same boundary conditions:

$$G\left(\vec{x},\vec{z}\right) = 0, \qquad \vec{x} \in \partial\Omega$$

then using the **infinite space version**: TODO

□

10 Method of Characteristics

This section covers an alternative, *powerful* way of solving **"PDEs"**, including some non-linear/quasi-linear ones (actually we end up solving ODEs).

10.1 Introduction

Before, we have discussed numerous methods to solve the following **PDE**:

$$u_{tt} = c^2 u_{xx}$$

Now, we will use the following method:

1. separate it into two **PDEs** without separation of variables
2. solve the two PDEs using **method of characteristics**

Theorem 10.1: Method of Characteristics

Given an **first order PDE**:

$$w_t + cw_x = 0$$

and having **no B.C.** but an initial condition of:

$$w(x, 0) = P(x)$$

Then, using the following trick:

$$w = w(x(t), t), \qquad x(t) \text{ being AN characteristic curve}$$

we get simplification to:

$$\begin{cases} \frac{dx}{dt} = c; & \text{characteristic curve} \\ \frac{d}{dt} w(x(t), t) = 0; & w \text{ along a characteristic curve } x(t) \end{cases}$$

and the solution is simply:

$$x(t) = ct + x_0$$
$$w(x_{(t)}, t) = P(x - ct).$$

where:

- the "initial condition" for the characteristic curve x_0 **determined which curve we are on**, i.e. which **single observer**
- the total solution graph for x, t would **combine the characteristic curves**, i.e. **combine the observers at** $x_{0_0}, x_{0_1}, \ldots$ **for the entire x-domain**

Proof. According to the above two steps, we do:

1. *separate the PDEs* into two **first order PDEs** using factoring:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0$$
$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

notice that the above two is **equivalent** to the original PDE:

$$u_{tt} - c^2 u_{xx} = 0$$

Now, we proceed by letting:

$$w = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u$$
$$v = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u.$$

and substituting into the **two factored PDEs respectively**, we get **two first order PDEs**:

$$w_t + cw_x = 0$$
$$v_t - cv_x = 0.$$

Reminder:

> *Recall that this was the **advection equation**, where solutions where the **Gaussian Curves** translating (see subsection 1.3)*

Now, we can proceed to the **method of characteristics**

2. Now, consider one of the first order PDE:

$$w_t + cw_x = 0$$

with **no boundary** and an **initial condition** of:

$$w(x, 0) = P(x)$$

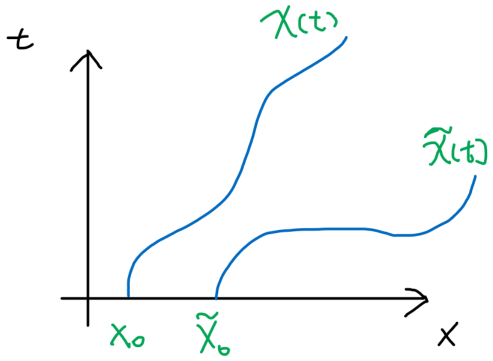
for some function $P(x)$.
Now, the crucial step is to consider an **characteristic curve** $x(t)$ such that:

$$w = w\left(x(t), t\right)$$

which means **how w evolves along the characteristics curve $x(t)$**

- an analogy could be $x(t)$ represents a moving observer.

The characteristic curves might look like:



where some feature needs to be satisfied are:

- the characteristic curves are **unique/distinct**, so that they do not cross each other
- uniqueness dependents on initial condition x_0 of the character sic curve

Therefore, we conclude that $x(t)$ **must be a solution to an ODE** with some initial condition $x(0) = x_0$.

Now, going back to our **first order PDE**, we see something interesting:

$$\frac{d}{dt} w\left(x(t), t\right) = 1 \cdot w_t + \frac{dx}{dt} \cdot w_x = w_t + \frac{dx}{dt} w_x$$

Now, we can **assume/specify the ODE for** $x(t)$ to be:

$$\frac{dx}{dt} = c$$

so that we can further get:

$$\frac{d}{dt} w\left(x(t), t\right) = w_t + \frac{dx}{dt} w_x = w_t + c w_x$$

notice that the RHS is **exactly our first order PDE**, therefore:

$$\frac{d}{dt} w\left(x(t), t\right) = 0$$

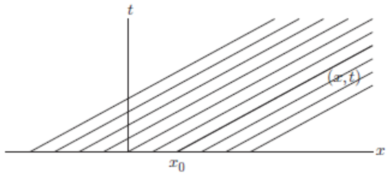
In total, we now simplified the problem to:

$$\begin{cases} \frac{dx}{dt} = c; & \text{characteristic curve} \\ \frac{d}{dt} w(x(t), t); & w \text{ along a characteristic curve } x(t) \end{cases}$$

solving the characteristic curve, we found that:

$$x(t) = ct + x_0$$

each curve is **unique for its initial condition** :



Now, solving for $w(x(t), t)$, we know that:

$$\frac{d}{dt}w = 0$$

so w is constant **along an characteristic curve**. Additionally, from **initial condition**, I know that:

$$w(x_0, 0) = P(x_0)$$

therefore, this means that:

$$w(x(t), t) = P(x_0)$$

for **AN characteristic curve starting at** x_0. Lastly, since we know $x(t) = ct + x_0$, we can **trace back from any given characteristic curve to its origin at** $t = 0$ **so that**:

$$x_0 = x(t) - ct$$

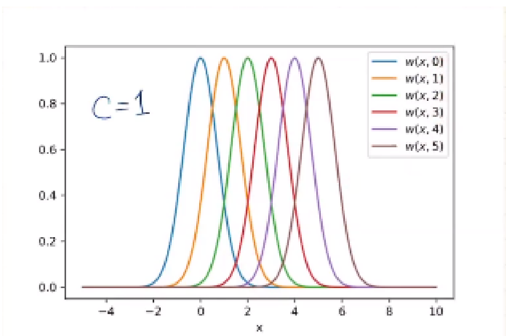
This would have solved the problem:

$$w(x(t), t) = P(x - ct)$$

which basically shows that we will have an **propagating initial condition to the right with speed** c

□

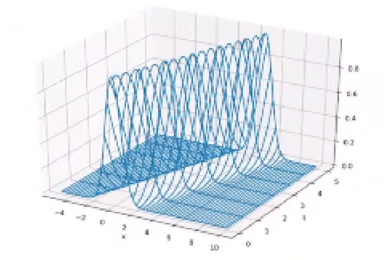
Graphically, we see that:

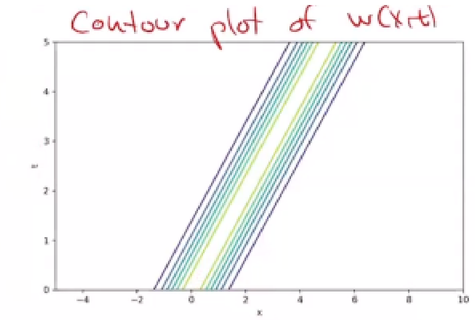


where:

- the initial condition is basically the Gaussian Curve
- this is exactly what we had for **advection equation**

Then, **along each characteristic curve**:





where we notice that:

- along each curve, w is constant
- w is **different for different curves**
- a snapshot of the solution can be seen by **combining the x-contributions of the w on the characteristic curves at a given time**
- the **symmetry of the contour plot** implies the **Gaussian Solution**
- the Solution Gaussian moves forward in x as time increases as **indicated by the contour plot for each horizontal time slice**

Note:

> Recall that we have only solved part of the entire PDE:
>
> $$u_{tt} - c^2 u_{xx} = 0$$
>
> solving the other half is trivial as we have done it already:
>
> $$v_t - cv_x = 0$$

> using the exact same **method of characteristic** of the above, we get:
>
> $$\begin{aligned} x(t) &= -ct \\ v(x(t), t) &= Q(x + ct). \end{aligned}$$
>
> for an initial condition of $v(x, 0) = Q(x)$.

Therefore, just for the sake of **completing the PDE**, we put everyhing back to assemble u. Recall that:

$$\begin{aligned} w &= u_t - cu_x \\ v &= u_t + cu_x. \end{aligned}$$

Therefore, we know that:

$$\begin{cases} u_t = \frac{1}{2}(w + v) = \frac{1}{2}\left[P(x - ct) + Q(x + ct)\right] \\ u_x = \frac{1}{2c}(v - w) = \frac{1}{2c}\left[Q(x + ct) - P(x - ct)\right] \end{cases}$$

Now, we can **define**:

$$F' = -\frac{1}{2c}P, \qquad G' = \frac{1}{2c}Q$$

so that the **following is true**:

$$u(x, t) = F(x - ct) + G(x + ct)$$

where you can **check this by taking the derivatives** explictly to see that it satisfies the u_t, u_x ODEs. This is also called the D'Alembert's Solution.

10.2 Method of Characteristics with Source and Damping

Now, we are kind of familiar with the trick. The general steps are simple.

Reminder:

General step for using this trick involve:

1. *change the* $v(x,t) \to v(x_{(t)}, t)$
2. *specify the characteristic curve equation according to the given first order PDE*
$$\frac{dx}{dt} = ...$$
3. *simply the PDE to an* ***ODE*** *for the unknown using a* ***total/material derivative****:*
$$\frac{d}{dt}v(x_{(t)}, t) = ...$$
4. *solve each one of them.*

Consider the problem:

$$v_t - cv_x + \alpha v = \beta$$

where we have:

- damping term αv
- source term β

and we have **no B.C.**, but an initial condition:

$$v(x, 0) = P(x)$$

Theorem 10.2: Solution

Using the above setup and the trick:

$$\frac{dx}{dt} = -c$$
$$\frac{d}{dt}v(x(t), t) + \alpha v = \beta.$$

becomes solving **ODEs**, and the solutions are:

$$x = -ct + x_0$$

$$
\begin{aligned}
v(x,t) &= \frac{\beta}{\alpha} + \left(P(x_0) - \frac{\beta}{\alpha}\right) e^{-\alpha t} \\
&= \frac{\beta}{\alpha}\left[1 - e^{-\alpha t}\right] + P(x+ct)e^{-\alpha t}.
\end{aligned}
$$

Proof. Taking the general routine, we do:

1. consider $v = v(x(t), t)$ along some characteristic curve
2. then, let the **characteristic curve ODE** be:

$$
\frac{dx}{dt} = -c, \qquad x(t) = -ct + x
$$

   notice that this choice will make the **below becomes**

3. the PDE becomes an **ODE** so that:

$$
\frac{d}{dt} v(x(t), t) + \alpha v = \beta
$$

   which is thanks to our characteristic curve.

4. Now, since the two equations are **independent**, it doesn't really matter which one first to solve. We have already solve the $x(t)$, so we now solve v.
   Notice that this ODE is solved by **integrating factors**:

$$
e^{-\alpha t}\left(e^{\alpha t} v\right)' = \beta
$$

   so the solution becomes:

$$
v = \frac{\beta}{\alpha} + ce^{-\alpha t}
$$

   for some constant c. This constant can be solved using **initial condition**:

$$
v(x_0, 0) = P(x_0)
$$

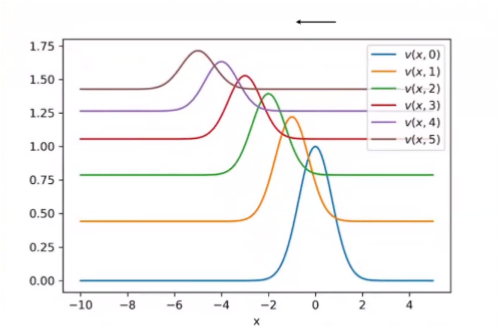
   Therefore:

$$
\begin{aligned}
v(x_0, 0) = \frac{\beta}{\alpha} + c = P(x_0) \\
c = P(x_0) - \frac{\beta}{\alpha}.
\end{aligned}
$$

   Therefore, we have the solution:

$$
\begin{aligned}
v(x,t) &= \frac{\beta}{\alpha} + \left(P(x_0) - \frac{\beta}{\alpha}\right) e^{-\alpha t} \\
&= \frac{\beta}{\alpha}\left[1 - e^{-\alpha t}\right] + P(x+ct)e^{-\alpha t}.
\end{aligned}
$$

   which we basically used our characteristic curve $x(t) = -ct + x_0$ for c being the **speed of light**. Additionally, as $t \to \infty$, we will just have $\frac{\beta}{\alpha}$:



where:

- the damping + source term is caused by αv, β, respectively:

$$v_t - cv_x + \alpha v = \beta$$

- curve moves to the left because we have $x + ct$

$\square$

10.3 Non-linear Characteristic Curve

All of the above had the similar characteristic curve:

$$x = \pm ct + x_0$$

Consider this example:

$$w_t + 3tw_x = w$$

with **no B.C.**, but an I.C. of:

$$w(x, 0) = f(x)$$

Theorem 10.3: Solution

Using the above setup and the characteristic equation:

$$\frac{dx}{dt} = 3t$$

we get the solution is

$$w(x, t) = f(x_0)e^t = f\left(x - \frac{3}{2}t^2\right)e^t$$

Proof. Doing again the same steps:

1. covert $w = w(x_{(t)}, t)$
2. consider the appropriate **characteristic equation**. In this case, we need at take:
$$\frac{dx}{dt} = 3t$$
then, the solution is:
$$x = \frac{3}{2}t^2 + x_0$$
being our **characteristic curves**
3. now the PDE should have been implied to an ODE:
$$\frac{d}{dt}w\left(x_{(t)}, t\right) = w_t + 3tw_x$$
the RHS was exactly our PDE, so we get:
$$\frac{dw}{dt} = w$$
being the ODE.

4. now, this is trivial to solve. The solution is:

$$w = ce^t$$

for a constant c, which can be solved using **initial condition**:

$$w(x_0, 0) = c = f(x_0)$$

now, again **walking backwards from the characteristic curve**, we get:

$$x_0 = x - \frac{3}{2}t^2$$

so I get the entire solution being:

$$w(x,t) = f(x_0)e^t = f\left(x - \frac{3}{2}t^2\right)e^t$$

which is **again an travelling initial condition**.

□

However, for a more general case, if we have:

$$w_t + c(x,t)w_x + \alpha(x,t)w = \beta(x,t)$$

we would have:

$$\frac{dx}{dt} = c(x,t)$$

which should be **solvable**, but the **uniqueness only holds for small** t. This means that

$$\frac{d}{dt}w\left(x_{(t)}, t\right) + \alpha(x,t)w = \beta(x,t)$$

is technically also a linear ODE, but if two characteristic curve coincide, we have a problem. Therefore for cases like this, we need to **take some extra care**, or use one of our **previous methods**.

10.4 Quasilinear PDE

Now, consider the case if we need to solve something like:

$$u_t + cu_x = Q$$

then, we tried the **method of characteristics**

$$\begin{cases} \frac{dx}{dt} = c(x, t, u) \\ \frac{d}{dt} u(x_{(t)}, t) = Q \end{cases}$$

where we see that the first equation is **coupled** to the second equation. However the PDE is still linear in u_x, u_t, so we can say that it is a **quasilinear system**.

10.4.1 Traffic Equation

Theorem 10.4: Modeeling Traffic in 1D

Consider a density of cars in 1D being $\rho(x, t)$, and the flux towards right for cars being q, then we can arrive at the conservation equation:

$$\rho_t + q_x = 0$$

Now, by **modelling the flux q** to be

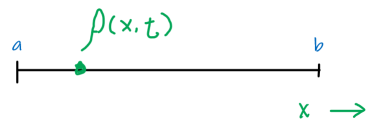
$$q = u(\rho) \cdot \rho = q(\rho)$$

for u representing the **velocity of the car**. Then, we can arrive that the PDE:

$$\rho_t + c(\rho)\rho_x = 0$$

for $c(\rho) = \frac{dq}{d\rho}$.

Proof. Start with the density representing **number of cars per unit length at a location x at time t**:



Then we know that the total number of cars in this segment is:

$$\int_a^b \rho(x, t)\, dx$$

Now, the typical starting point is to consider the **flux of cars (to the right) being $q(a, t)$ at $x = a$, and $q(b, t)$ at $x = b$.** Therefore, I have:

$$\frac{d}{dt} \int_a^b \rho(x, t)\, dx = q(a, t) - q(b, t).$$

Then, we can rewrite RHS as:

$$q(a,t) - q(b,t) = -\int_a^b \frac{\partial}{\partial x} q \, dx$$

Therefore, we got the **conservation**:

$$\int_a^b \left(\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x} q \right) dx = 0$$

Because this must be true for any segment, this means that:

$$\rho_t + q_x = 0$$

Now, recall that the **unknown of interest is** ρ. So we need to find a way to represent q in a related way. Consider:

1. $q = u\rho$, for meaning the **flux of cars to the right** is proportional to the **number of cars** ρ **at that** x and the **speed of those cars**.
2. $u \approx u(\rho)$ where we assume that the **speed of cars** only depend on the **density**. (In reality, there could be x and t dependence as well, but they are ignored here)

Now, using the simple model of:

$$q = u(\rho) \cdot \rho = q(\rho)$$

our conservation equation becomes:

$$\rho_t + q_x = \rho_t + \frac{dq}{d\rho}\rho_x = \rho_t + c(\rho)\rho_x = 0.$$

So we got our PDE with **coupled coefficient** and the traffic equation. □

Note:

> Observe that $c(\rho)$ representing the wave speed is then:
>
> $$c(\rho) = \frac{dq}{d\rho} = u(\rho) + u'(\rho) \cdot \rho$$
>
> which means that $c(\rho) \neq u(\rho)$, i.e. **speed of wave is DIFFERENT from speed of cars**

Now, we want to attempt to solve it.

Theorem 10.5: Solution to 1D Traffic Flow

First, we also need to have the expression of $u(\rho)$. We can consider:

$$u(\rho) = u_{\max} \left(1 - \frac{\rho}{\rho_{\max}} \right)$$

so that we have:

- $u(0) = u_{\max}$ for modeling **light traffic**

- $u(\rho_{\max}) = 0$ for modeling **heavy traffic**

Then the **method of characteristic gives**:

$$\begin{cases} \frac{d}{dt}x = c(\rho), & \text{I.C.}{=}x(0) = x_0 \\ \frac{d}{dt}\rho(x_{(t)}, t) = 0, & \text{I.C.}{=}\rho(x_{(0)}, 0) = \rho_0(x_0) \end{cases}$$

Then, suppose we have $\rho_{\max} = 10, u_{\max} = 5$, and the initial condition is:

$$\rho_0(x) = \begin{cases} 3, & x < 5 \\ 1, & x \geq 5 \end{cases}$$

Then the solution is:

$$\rho(x_{(t)}, t) = \rho(x_{(0)}, 0) = \rho_0(x_0), \qquad \text{along a characteristics}$$

And **each characteristic curve** is described by:

$$x(t) = c(\rho_0\,(x_0))t + x_0$$

Combining both will give the solution plot.

$$\rho(x_{(t)}, t) = \rho_0(x_0) = \rho(x - c_{(\rho_0(x_0))}, t), \qquad \text{along a characteristics}$$

Proof. Consider the model we mentioned for $u(\rho)$ being the **velocity of cars**:

$$u(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$$

so that we have:

- $u(0) = u_{\max}$ for modeling **light traffic**
- $u(\rho_{\max}) = 0$ for modeling **heavy traffic**

Then, we have our q:

$$q(\rho) = u(\rho) \cdot \rho = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)\rho$$

then, since we also had:

$$c(\rho) = \frac{dq}{d\rho} = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)$$

We can now **treat the PDE with method of characteristics**:

$$\rho_t + c(\rho) \cdot \rho_x = 0$$

And we would be dealing with:

$$\begin{cases} \frac{d}{dt}x = c(\rho), & \text{I.C.}{=}x(0) = x_0 \\ \frac{d}{dt}\rho(x_{(t)}, t) = 0, & \text{I.C.}{=}\rho(x_{(0)}, 0) = \rho_0(x_0) \end{cases}$$

Now, we **start our method of characteristics**:

1. since the first equation is coupled to the second, we **start with solving the second equation**. Since:

$$\frac{d}{dt}\rho = 0, \qquad \text{constant along characterstics}$$

This means that:

$$\rho(x_{(t)}, t) = \rho(x_{(0)}, 0) = \rho_0(x_0), \qquad \text{along a characteristics}$$

2. Now, I know ρ along a characteristic curve, I can solve **the first equation**:

$$\frac{dx}{dt} = c(\rho), \qquad \text{for a characteristic curve with } x(0) = x_0$$

Therefore, since **on a characteristic curve, ρ is constant**, I simply have:

$$\frac{dx}{dt} = c(\rho\left(x_{(t)}, t\right)) = c(\rho\left(x_{(0)}, 0\right)) = c(\rho_0(x_0))$$

Hence **each characteristic curve** is described by:

$$x(t) = c(\rho_0\left(x_0\right))t + x_0$$

where basically the **gradient c is dependent on the initial density**.

3. Therefore, we again obtain the **moving initial condition**:

$$\rho(x_{(t)}, t) = \rho_0(x_0) = \rho_0(x - c_{(\rho_0(x_0))}t), \quad \text{along a characteristics at } x_0$$

Though now we should be able to plug in I.C. and got the solution, but notice the **risk of curves crossing each other** here. Suppose two curves do cross at $t = t^*$, for each starting at $x_0, x_1; x_1 > x_0$then:

$$c(\rho_0(x_0))t^* + x_0 = c(\rho_0(x_1))t^* + x_1$$
$$t^* = \frac{x_1 - x_0}{c(\rho_0(x_0)) - c(\rho_0(x_1))}.$$

Therefore, since we already know that $c(\rho)$ is a **decreasing function of** ρ, so if $\rho_0(x)$ is **decreasing as x increases**, then we would have $t^* < 0$ and **no curve crossing**!

Note:

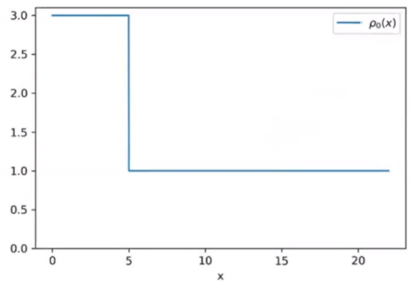
> In this example, we will discuss the case for $\rho_0(x)$, the initial condition, be an decreasing function of x, so that it means $c(\rho)$ is higher for larger x, which means **fast cars are in the front**.
> The other case will be discussed in the **next section**

---

4. Now, we solve the equation with some initial condition. Consider the setup of:

$$\rho_0(x) = \begin{cases} 3, & x < 5 \\ 1, & x \geq 5 \end{cases}$$

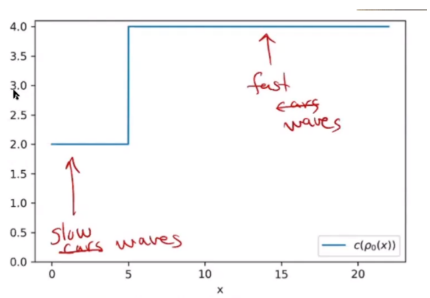
which graphically looks like:



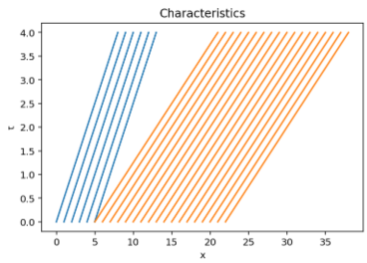
Then, we can compute the $c(\rho) = c(\rho_0(x_0))$ needed by the characteristic curves:

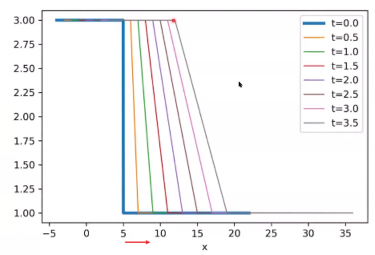
$$c(\rho_0(x)) = \begin{cases} 2, & x < 5 \\ 4, & x \geq 5 \end{cases}$$

using $\rho_{\max} = 10, u_{\max} = 5$. The graph looks like this:



where fast cars/waves with higher $c(\rho)$ is in the front.
Then, our characteristic curves are ready, and our solution:





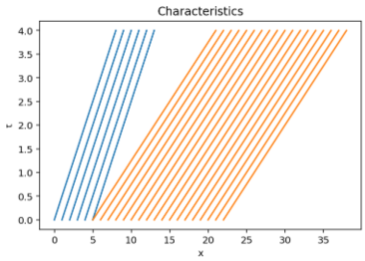
where:

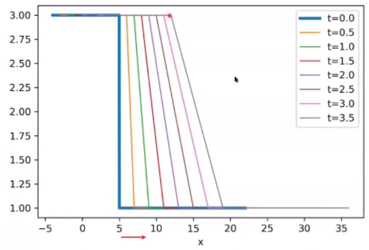
- $\rho = \rho_0$ being constant **long a characteristic curve**
- if we take a **horizontal slice of the characteristic curves**, we get the solution ρ at a some fixed time t_0
- the only problem left is the **gap between the discontinuities**, which is also indicated by the **gap between the characteristics curve for blue and orange lines**. Sometimes, this gap can be understood as a **rarefaction of waves**.

□

10.4.2 Gap Between Characteristic in Traffic Flow

In the case of the **fast cars in front**, we see the following characteristics:





where:

- the **gap** between blue and orange lines are assumed to be a **linear relation for ρ** in the solution plot (i.e. linear line connecting discontinuities)

This section aims to explain why the **linear choice**.

Proof. The first and main step is to make an **assumption**: the value of ρ only **depends on the gradient of the characteristics**, $\rho = \rho\,(x/t)$, because:

- solution ρ is **constant** along a characteristic curve
- solution ρ is **different for different characteristic curves**, but the gradient $\frac{x}{t}$ is **also different for different characteristic curves**

Therefore, such assumption is somewhat sensible. Now, if we consider:

$$\xi = \frac{x}{t} \to \rho(x,t) = \rho(\xi)$$

Then preparing for some derivatives:

$$\frac{d\xi}{dx} = \frac{1}{t}, \qquad \frac{d\xi}{dt} = -\frac{x}{t^2}$$

Therefore, we can **rewrite the PDE** to become:

$$\begin{aligned}
\rho_t + c(\rho)\rho_x &= \frac{d\xi}{dt}\frac{d\rho}{d\xi} + c(\rho)\frac{d\xi}{dx}\frac{d\rho}{d\xi} \\
&= -\frac{x}{t^2}\rho_\xi + c(\rho)\frac{1}{t}\rho_\xi \\
&= \left[c(\rho) - \frac{x}{t}\right]\rho_\xi \\
&= 0.
\end{aligned}$$

This means that we have a **similar case for characteristics**:

$$\rho(\xi) = 0, \qquad \rho \text{ is constant for the same } \frac{x}{t}$$

which means that ρ **is constant along characteristics!** Same conclusion before.
However, the difference is:

$$c(\rho) = \frac{x}{t}, \qquad \text{wave speed}$$

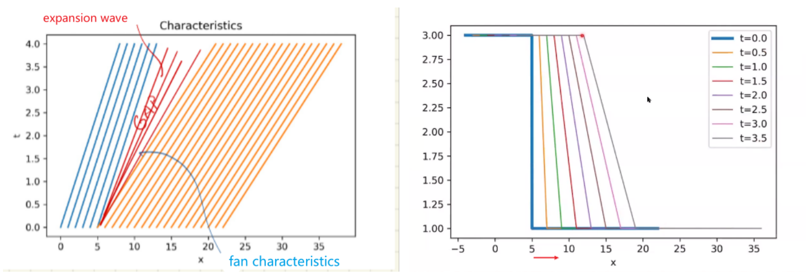
Since we know the expression for wave speed, we have:

$$\begin{aligned} c(\rho) &= u_{\max}\left[1 - \frac{2\rho}{\rho_{\max}}\right] \\ &= \frac{x}{t}. \end{aligned}$$

we have **another expression for ρ**:

$$\rho = \frac{\rho_{\max}}{2}\left[1 - \frac{x}{t}\frac{1}{u_{\max}}\right]$$

which means that ρ **is indeed only dependent on** $\frac{x}{t}$, and that ρ **is linear with gradient $\frac{x}{t}$ of the characteristics!**
Therefore, graphically we see that



where is linear characteristics in the gap is also called the **fan characteristics**.

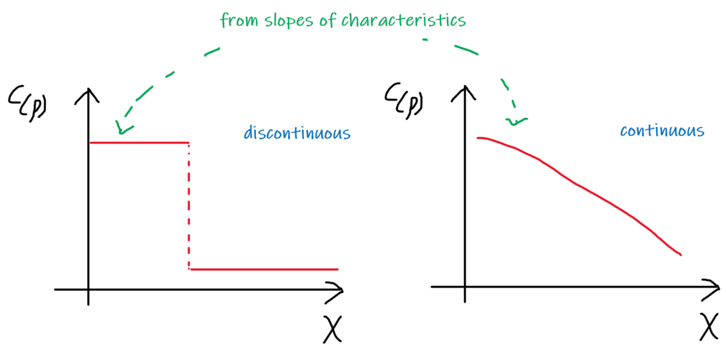
□

10.4.3 Shock Waves for Traffic Flow

Now, we consider the case where the initial condition will cause **characteristics curves crossing**. This, as you will see in the future, is **common for non-linear/quasi-linear PDEs**, and that you will also see:

- solution depends **critically**/varies greatly for small changes in initial condition

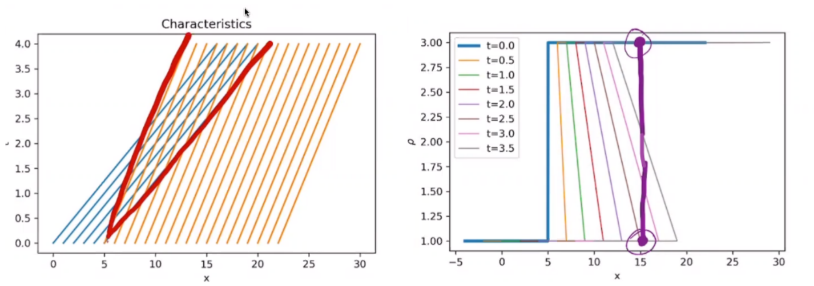
Consider the case where we have the I.C. being ρ_0 being a **increasing function of** x, so that the speed is:



where we see that:

- now the I.C. gives **slow cars are in the front**!
- both continuous and discontinuous case will be examined so we confirm that this issue has **nothing to do with discontinuities**

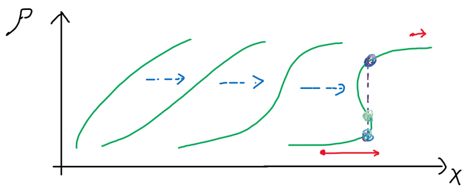
Therefore, now we have **collisions of characteristics curves**, so we have:



where:

- because characteristics cross, this means that for some (x,t), we have a **two possible values for** ρ (the purple line)
- this issue can be also seen if you **walk along a characteristics curve at** $x_0 = 8$**, for example**. You will meet two choices for ρ from the same initial condition $x_0 = 8$.

For the continuous case, the situation is even worse:

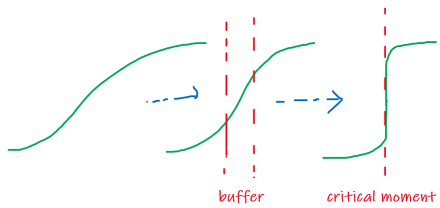


where:

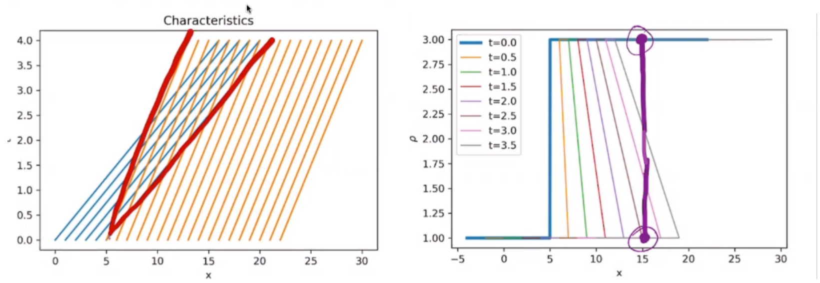
- bottom part is **faster than top**
- now, we could have **three possible values** ρ of the same (x, t).

---

resolution. If we consider what is happening step by step in time, we realize some **critical moment** when things go wrong:

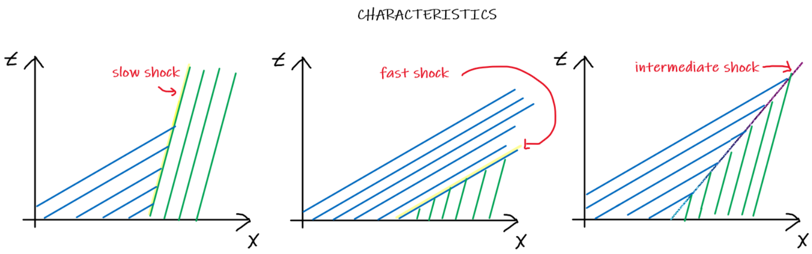


this means that essentially, we reconsidering the same problem in the **discontinuous case**, which is just the **critical moment**:



The big picture of resolution means that, at the **critical point**, we should have cars not colliding by making them having the **same speed**.

This means the following three possible scenarios:



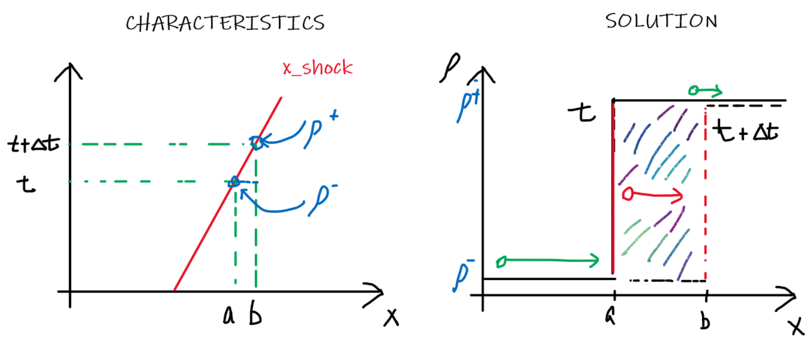
where:

- for the solution plot ρ, the difference will be at **what speed** should the infinite gradient part be travelling at

Note:

> Note, in some cases, you might end up with **new** characteristics emitting out from the shock wave (if we have steep gradient on the left and flat gradient on the right). This in general will **violate conservation law**, as you shall see soon.

The resolution in this case comes from considering **conservation**:



where notice that:

- after time Δt, we have **lost some number of cars in the shaded region** due to the shock wave (i.e. the **total number of cars changed**)

Consider the region between $x \in [a, b]$. We then know that the **total number of cars would be**:

$$N(t) = \int_a^b \rho(x, t)\, dt$$

and since it changes, we have **flux** q **being defined towards the right**:

$$\frac{dN}{dt} = q(a) - q(b)$$

However, notice that from the **characteristics curve**, we see that:

$$\begin{aligned} N(t) &= (b-a)\,\rho^- = (x_s\,(t+\Delta t) - x_s\,(t))\,\rho^- \\ N(t+\Delta t) &= (b-a)\,\rho^+ = (x_s\,(t+\Delta t) - x_s\,(t))\,\rho^+ \\ &. \end{aligned}$$

Therefore, applying conservation equation:

$$\begin{aligned} \frac{N(t+\Delta t) - N(t)}{\Delta t} &= \frac{x_s(t+\Delta t) - x_s(t)}{\Delta t}\left[\rho^+ - \rho^-\right] \\ \frac{dN}{dt} &= \frac{dx_s}{dt}\left[\rho^+ - \rho^-\right] \\ &= q(a) - q(b). \end{aligned}$$

Therefore, we obtained the **gradient of shock wave** $\rightarrow$ **speed of "intermediate cars"** being:

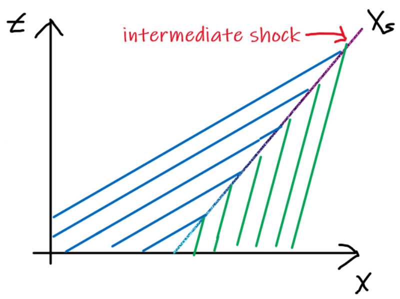
$$\frac{dx_s}{dt} = \frac{q(a) - q(b)}{\rho^+ - \rho^-} = \frac{q(\rho^-) - q\left(\rho^+\right)}{\rho^+ - \rho-}.$$

This is also called the **RankineHugoniot Jump Condition**, which is also considered the **Entropy Condition** in relevant studies such as Physics. Then, you will generally find that:

$$c(\rho^+) < \frac{dx_s}{dt} < c(\rho^-)$$

So the correct graph is actually the **intermediate one**

CHARACTERISTICS



and from this x_s, you get the speed of those **squashed together cars**, and you can plot your ρ with the shock. □

11 Appendix

11.1 Common Solutions from ODE

Below are some common ODEs with their solution/ansatz:

1. Simple case
$$x'(t) = a(t)x(t) + f(t)$$
   - *goes to* $x(t) = e^{\phi(t)}(\int e^{-\phi(t)} f(t)dt + c)$,
   - for $\phi = \int a(t)dt$
   - **derived with integrating factor**: $\frac{d}{dt}(e^{-\phi(t)}x(t)) = e^{-\phi(t)}(x'(t) - \phi'(t)x(t)) = e^{-\phi(t)}f(t)$
2. with constant coefficients:
$$ax'' + bx' + cx = 0$$
   - *ansatz* $x = Ae^{rt}$
   - **derived by plug in and solve for** r, which looks like $ar^2 + br + c = 0$, $r_\pm = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
   - for example, if $b^2 > 4ac$, we get $x = A_1 e^{r_+ t} + A_2 e^{r_- t}$
3. solve for $\varphi(x)$
$$x^2\varphi'' + \alpha x\varphi' + \beta\varphi = 0$$
   - *ansatz* $\varphi = Ax^r$
   - **derived by plug in and solve for** r, which looks like: $r(r-1) + \alpha r + \beta = 0$